Gödel's Incompleteness Theorem

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Winter 2011

Left Over Homework

- Exercise 3 in Chapter 3, that is

  [...] We let $\mathsf{Seq}_2(x)$ *denote that* x *is sequence number. We let* $(x, y) \in z$ *denote that the pair* (x, y) *is a member of the sequence, numbered by* z*. Finally let* $(x_1, y_1) \prec_z (x_2, y_2)$ *denote that* (x_1, y_1) *occurs in* z *before* (x_2, y_2)*.*

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- Exercise 5 in Chapter 3, that is:

  [...] Let $M(x, y, z)$ *be the relation "*E_x *is substitutable for* E_y *in* E_z*" and show that this is Arithmetic.*

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- Exercise 6 in Chapter 3, that is:

  [...] Show that the set of Gödel numbers of the axioms of L_5' *is Arithmetic.*

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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Σ_0-relations

Definition

an atomic Σ_0-formula is a formula of the form

$$s = t \qquad s + t = u \qquad s \cdot t = u \qquad s \leqslant t$$

where s, t, u are variables or numerals

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Definition

the Σ_0-formulas are defined inductively:

1. every atomic Σ_0-formula is a Σ_0-formula
2. if A, B are Σ_0-formulas, v_i a variable, t a numeral or variable $\neq v_i$, then

$$\neg A \qquad A \rightarrow B \qquad \forall v_i(v_i \leqslant t \rightarrow A)$$

   are Σ_0-formulas

Convention

- as before we write $A \wedge B$, $A \vee B$, $(\forall v_i \leqslant t)\ A$ as abbreviations of

$$\neg(A \to \neg B) \qquad \neg A \to B \qquad \forall v_i (v_i \leqslant t \to A)$$

- we write $(\exists v_i \leqslant t)\ A$ as abbreviation for

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Definition

1. the quantifiers $\exists v_i \leqslant t$ and $\forall v_i \leqslant t$ are called bounded quantifiers
2. a relation is a Σ_0-relation if expressible by a Σ_0-formula
3. Σ_0-relations are called constructive arithmetic relations

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Fact

truthhood of Σ_0-sentences is decidable

Definition

1. a Σ_1-formula is a formula of the form

$$\exists v_{n+1} F(v_1, \dots, v_n, v_{n+1})$$

   where $F(v_1, \dots, v_n, v_{n+1})$ is a Σ_0-formula

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Definition

we inductively define the class of Σ-formulas

1. every Σ_0-formula is a Σ-formula
2. if A, B are Σ-formula, v_i a variable, then $A \vee B$, $A \wedge B$, and $\exists v_i A$ are Σ-formulas
3. if A is a Σ_0-formula and B a Σ-formula, then $A \to B$ is a Σ-formula
4. if A is a Σ-formula, v_i, v_j a distinct variables, and $\overline{n}$ a numeral

$$(\exists v_i \leqslant v_j)A \qquad (\forall v_i \leqslant v_j)A \qquad (\exists v_i \leqslant \overline{n})A \qquad (\forall v_i \leqslant \overline{n})A$$

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- *the Σ-relations are exactly the Σ_1-relations*
- *let $M = \{n \mid P(\overline{n})\}$, where P is a Σ-relations; then M is recursively enumerable*

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Fact

the relation $x < y$ is Σ_0, as $x < y$ holds iff $x \leqslant y \wedge x \neq y$; hence we can make use of the bounded quantifiers $\exists x < t$ and $\forall x < t$

Concatenation to a Prime Basis

Lemma

for any prime number p, the following conditions is Σ_0

1. $x \operatorname{div} y$, *that is,* $x \mid y$
2. $\mathrm{Pow}_p(x)$, *that is x is a power of p*
3. $y = p^{|x|_p}$, *that is y is the smallest positive power of $p \geqslant x$*

Proof.

on the whiteboard ∎

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Lemma

for any prime p, the relation $x *_p y = z$ is Σ_0

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for any prime p, the following relations are Σ_0*:*

1. xB_py, xE_py, *and* xP_py
2. $\forall\ n \geqslant 2$: $x_1 *_p x_2 *_p \cdots *_p x_n = y$
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Definition

the axiom system PA is defined as $\mathrm{PA} = \mathrm{PE} - \{\mathrm{exp}\}$

Exponentiation is arithmetic

Lemma (The Finite Set Lemma)

$\exists$ a Σ_0-relation $K(x, y, z)$ such that

1. *$\forall$ finite sequences $(a_1, b_1), \ldots, (a_n, b_n)$ of pairs of natural numbers $\exists\ z \in \mathbb{N}$ such that $\forall\ x, y \in \mathbb{N}$, $K(x, y, z)$ holds iff $(x, y) = (a_i, b_i)$ for some $i \in \{1, \ldots, n\}$*
2. *if $K(x, y, z)$ holds, then $x, y \leqslant z$*

Exponentiation is arithmetic

Lemma (The Finite Set Lemma)

$\exists$ *a* Σ_0*-relation* $K(x, y, z)$ *such that*

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2. *if* $K(x, y, z)$ *holds, then* $x, y \leqslant z$

Theorem

the relation $x^y = z$ *is* Σ_1

Proof.

on the whiteboard using the above lemma ∎

Proof of The Finite Set Lemma

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we identify numbers with their base 13 representation

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- $1(x)$ denotes that $x = 1 \cdots 1$, $1(x)$ is Σ_0

$$1(x) :\Leftrightarrow x \neq 0 \wedge (\forall y \leqslant x)(yPx \rightarrow 1Py)$$

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- let $\theta = ((a_1, b_1), \ldots, (a_n, b_n))$ and let f be the any frame which is longer than any frame that is part of any of the numbers in θ, then a sequence number of θ is

$$ffa_1 fb_1 ff \cdots ffa_n fb_n ff$$

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the frame f plays the role previously played by δ

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Definition

we define the relation $K(x, y, z)$:

$$(\exists w \leqslant z)(w \text{ mf } z \wedge wwxwywwPz \wedge \neg(wPx) \wedge \neg(wPy))$$

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Corollary

the system PA *is incomplete*