Gödel's Incompleteness Theorem

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Homework

- Chapter IV, Exercise 1, that is:

  [...] Since G is a true sentence, the system $\mathrm{PA} \cup \{G\}$ *is also a correct system. Is it complete?*

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

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More on Σ_1-Relations

Lemma

1. *any Σ_0-relation is Σ_1*

Proof.

on the whiteboard ∎

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2. *if* $R(x_1, \dots, x_n, y)$ *is* Σ_1*, then the following relation is* Σ_1*:*

$$\exists y R(x_1, \dots, x_n, y)$$

Proof.

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2. *if $R(x_1, \ldots, x_n, y)$ is Σ_1, then the following relation is Σ_1:*
$$\exists y R(x_1, \ldots, x_n, y)$$
3. *if $R_1(x_1, \ldots, x_n)$ and $R_2(x_1, \ldots, x_n)$ are Σ_1, then so are the relations:*
$$R_1(x_1, \ldots, x_n) \vee R_2(x_1, \ldots, x_n) \qquad R_1(x_1, \ldots, x_n) \wedge R_2(x_1, \ldots, x_n)$$

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4. *if R is Σ_0, S is Σ_1, then $R \to S$ is Σ_1*

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4. *if R is Σ_0, S is Σ_1, then $R \to S$ is Σ_1*
5. *if $R(x_1, \dots, x_n, y, z)$ is Σ_1, then so are the relations:*
$$(\exists y \leqslant z) R(x_1, \dots, x_n, y, z) \qquad (\forall y \leqslant z) R(x_1, \dots, x_n, y, z)$$

Proof.

on the whiteboard ∎

Lemma (revisited)

the Σ-relations are exactly the Σ_1-relations

Proof.

by induction on the degree of formulas representing the relations using the above lemma ∎

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Corollary

1. *if A is Σ_1, then so is A^**
2. *the sets $(\mathsf{P}_\mathsf{A})^*$ and $(\mathsf{R}_\mathsf{A})^*$ are Σ_1*

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Corollary

1. *if A is Σ_1, then so is A^**
2. *the sets $(\mathrm{P_A})^*$ and $(\mathrm{R_A})^*$ are Σ_1*

what is this?

Recursive Sets

Definition

- a set or relation R is called recursive if R and $\sim R$ is Σ_1
- a function $f(x_1, \ldots, x_n)$ is recursive if the relation $f(x_1, \ldots, x_n) = y$ is recursive

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Theorem

$\forall\ n \in \mathbb{N}$, $k \leqslant n$, *sequence* $(a_1, \ldots, a_k)$ *such that* $a_i \in K_{11}$ *and* $a_i \leqslant n$, *then we have:* $\delta a_1 \delta \ldots \delta a_k \delta \leqslant \pi(n)$

Proof.

on whiteboard ∎

Lemma (revisited)

let $M = \{n \mid P(\overline{n})\}$*, where* P *is a* Σ*-relations; then* M *is recursively enumerable*

Proof.

- let M be a Turing machine (TM)
- let α, β be configurations of a TM
- let $\xrightarrow[M]{n}$ denote the n-step relation of a TM and recall:

$$\alpha \xrightarrow[M]{*} \beta :\Leftrightarrow \exists n\ \alpha \xrightarrow[M]{n} \beta$$

- the relation $\alpha \xrightarrow[M]{n} \beta$ is recursive
- recall

$$\mathsf{L}(M) = \{x \in \Sigma^* \mid (s, \vdash x\sqcup^\infty, 0) \xrightarrow[M]{*} (t, y, n)\}$$

- the set $\mathsf{L}(M)$ is Σ_1

∎

Corollary

the system PA *is incomplete*

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Proof.

- $(\sim \mathrm{P_A})^*$ is arithmetic
- $\exists$ arithmetic formula $H(v_1)$ expressing $(\sim \mathrm{P_A})^*$
- let $h := \ulcorner H(v_1) \urcorner$ and let $H[\overline{h}]$ be the Gödel sentence of $(\sim \mathrm{P_A})^*$
- we obtain:

$$H[\overline{h}] \text{ holds} \iff h \in (\sim \mathrm{P_A})^* \iff d(h,h) \notin \mathrm{P_A} \iff$$
$$\iff H[\overline{h}] \text{ is not provable}$$

- as PA is correct $H[\overline{h}]$ cannot be provable
  otherwise $H[\overline{h}]$ would be false and provable
- hence $H[\overline{h}]$ is true, but not provable

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we consider an axiom system $\mathcal{S}$ over the language of PA such that

1. $\mathcal{S}$ includes axioms for first-order with equality
2. $\mathcal{S}$ has rules *modus ponens* and generalisation
3. in addition $\mathcal{S}$ has arbitrary non-logical axioms

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  $\mathcal{S} \vdash \exists w F(w)$ and $\mathcal{S} \vdash \neg F(\overline{0}), \dots, \mathcal{S} \vdash \neg F(\overline{n}), \mathcal{S} \vdash \neg F(\overline{n+1}), \dots$

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Fact

ω-consistency implies consistency

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PA is recursively axiomatisable

Definition

given two systems $\mathcal{S}_1$, $\mathcal{S}_2$ we say that $\mathcal{S}_1$ is a subsystem of $\mathcal{S}_2$ ($\mathcal{S}_2$ is an extension) of $\mathcal{S}_1$), if all provable formulas of $\mathcal{S}_1$ are provable in $\mathcal{S}_2$

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Theorem

if PA *is ω-consistent, then it is incomplete*

Towards Gödel's Incompleteness Proof

Theorem ①

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Definition

- $F(v_1)$ represents A if for all $n \in \mathbb{N}$: $F(\overline{n})$ is provable $\iff$ $n \in A$
- $F(v_1, \ldots, v_n)$ represents R if for all $(m_1, \ldots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \ldots, \overline{m}_n) \text{ is provable } \iff (m_1, \ldots, m_n) \in R$$

we also say that $F(v_1, \ldots, v_n)$ represents the relation $R(x_1, \ldots, x_n)$

let P denote the set of Gödel numbers of provable formulas in $\mathcal{S}$ and R the set of Gödel numbers of refutable formulas in $\mathcal{S}$

Lemma

for any formula $H(v_1)$ with Gödel number h

1. $H(\overline{h})$ *is provable in* $\mathcal{S}$ *iff* $h \in P^*$
2. $H(\overline{h})$ *is refutable in* $\mathcal{S}$ *iff* $h \in R^*$

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Theorem

1. *suppose* $\mathcal{S}$ *is consistent*
2. *the negation of* $H(v_1)$ *represents* P^* *in* $\mathcal{S}$
3. *let* $h := \ulcorner H(v_1) \urcorner$

then the sentence $H(\overline{h})$ *is neither provable or refutable in* $\mathcal{S}$

Proof.

on the whiteboard ∎

Corollary

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Theorem (a dual of the above theorem)

if R^* is representable in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete

Proof.

as above ∎

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Theorem (a dual of the above theorem)

if R^* is representable in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete

Proof.

as above ∎

Definition

a formula $F(v_1, v_2)$ enumerate a set A in $\mathcal{S}$ if $\forall n \in \mathbb{N}$:

1. if $n \in A$, $\exists m \in \mathbb{N}$ such that $\mathcal{S} \vdash F(\overline{n}, \overline{m})$
2. if $n \notin A$, $\forall m \in \mathbb{N}$ we have $\mathcal{S} \vdash \neg F(\overline{n}, \overline{m})$

a set A is enumerable if $\exists$ formula $F(v_1, v_2)$ that enumerates A

ω-consistency Lemma

Definition

a formula $F(v_1, \dots, v_n, v_{n+1})$ enumerate a relation $R(x_1, \dots, x_n)$ in $\mathcal{S}$ if $\forall n \in \mathbb{N}$:

1. if $R(k_1, \dots, k_n)$ holds, $\exists m \in \mathbb{N}$ such that $\mathcal{S} \vdash F(\overline{k_1}, \dots, \overline{k_n}, \overline{m})$
2. if $R(k_1, \dots, k_n)$ does not hold, $\forall m \in \mathbb{N}$ we have $\mathcal{S} \vdash \neg F(\overline{k_1}, \dots, \overline{k_n}, \overline{m})$

a relation $R(x_1, \dots, x_n)$ is enumerable if $\exists$ formula $F(v_1, \dots, v_n, v_{n+1})$ that enumerates $R(x_1, \dots, x_n)$

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a relation $R(x_1, \dots, x_n)$ is enumerable if $\exists$ formula $F(v_1, \dots, v_n, v_{n+1})$ that enumerates $R(x_1, \dots, x_n)$

Lemma (ω-consistency Lemma)

if $\mathcal{S}$ is ω-consistent, and if set A is enumerable by $F(v_1, v_2)$, then A is representable by $\exists v_2 F(v_1, v_2)$ in $\mathcal{S}$

Theorem

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Theorem

suppose $F(v_1, v_2)$ enumerate P^* in $\mathcal{S}$; let $f := \ulcorner \forall v_2 \neg F(v_1, v_2) \urcorner$ and let $G := \forall v_2 \neg F(\overline{f}, v_2)$, then:

1. *if $\mathcal{S}$ is consistent, then G is not provable in $\mathcal{S}$*
2. *if $\mathcal{S}$ is ω-consistent, then $\mathcal{S}$ is incomplete*

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Theorem (a dual of the above theorem)

suppose $F'(v_1, v_2)$ enumerate R^* in $\mathcal{S}$; let $f' := \ulcorner \exists v_2 F'(v_1, v_2) \urcorner$ and let $G' := \exists v_2 F'(\overline{f}, v_2)$, then:

1. *if $\mathcal{S}$ is consistent, then G' is not provable in $\mathcal{S}$*
2. *if $\mathcal{S}$ is ω-consistent, then $\mathcal{S}$ is incomplete*