Gödel's Incompleteness Theorem

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Homework

- Chapter IV, Exercise 1, that is:

  [...] Since G is a true sentence, the system PA $\cup \{G\}$ *is also a correct system. Is it complete?*

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

More on Σ_1-Relations

Lemma

1. *any Σ_0-relation is Σ_1*
2. *if $R(x_1, \ldots, x_n, y)$ is Σ_1, then the following relation is Σ_1:*
   $$\exists y R(x_1, \ldots, x_n, y)$$
3. *if $R_1(x_1, \ldots, x_n)$ and $R_2(x_1, \ldots, x_n)$ are Σ_1, then so are the relations:*
   $$R_1(x_1, \ldots, x_n) \vee R_2(x_1, \ldots, x_n) \qquad R_1(x_1, \ldots, x_n) \wedge R_2(x_1, \ldots, x_n)$$
4. *if R is Σ_0, S is Σ_1, then $R \to S$ is Σ_1*
5. *if $R(x_1, \ldots, x_n, y, z)$ is Σ_1, then so are the relations:*
   $$(\exists y \leqslant z) R(x_1, \ldots, x_n, y, z) \qquad (\forall y \leqslant z) R(x_1, \ldots, x_n, y, z)$$

Proof.

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Lemma (revisited)

the Σ-relations are exactly the Σ_1-relations

Proof.

by induction on the degree of formulas representing the relations using the above lemma ■

Corollary

1. *if A is Σ_1, then so is A^**
2. *the sets $(\mathrm{P_A})^*$ and $(\mathrm{R_A})^*$ are Σ_1*

Recursive Sets

Definition

- a set or relation R is called recursive if R and $\sim R$ is Σ_1
- a function $f(x_1, \dots, x_n)$ is recursive if the relation $f(x_1, \dots, x_n) = y$ is recursive

Lemma

we define $\pi(x) := 13^{x^2+x+1}$, then $\pi(x)$ is recursive

Theorem

$\forall\ n \in \mathbb{N}$, $k \leqslant n$, sequence $(a_1, \dots, a_k)$ such that $a_i \in K_{11}$ and $a_i \leqslant n$, then we have: $\delta a_1 \delta \dots \delta a_k \delta \leqslant \pi(n)$

Proof.

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Lemma (revisited)

let $M = \{n \mid P(\overline{n})\}$, where P is a Σ-relations; then M is recursively enumerable

Proof.

- let M be a Turing machine (TM)
- let α, β be configurations of a TM
- let $\xrightarrow[M]{n}$ denote the n-step relation of a TM and recall:
$$\alpha \xrightarrow[M]{*} \beta :\Leftrightarrow \exists n\ \alpha \xrightarrow[M]{n} \beta$$
- the relation $\alpha \xrightarrow[M]{n} \beta$ is recursive
- recall
$$\mathrm{L}(M) = \{x \in \Sigma^* \mid (s, \vdash x\sqcup^\infty, 0) \xrightarrow[M]{*} (t, y, n)\}$$
- the set $\mathrm{L}(M)$ is Σ_1

■

Corollary

the system PA is incomplete

Proof.

- $(\sim\mathrm{P_A})^*$ is arithmetic
- $\exists$ arithmetic formula $H(v_1)$ expressing $(\sim\mathrm{P_A})^*$
- let $h := \ulcorner H(v_1) \urcorner$ and let $H[\overline{h}]$ be the Gödel sentence of $(\sim\mathrm{P_A})^*$
- we obtain:
$$H[\overline{h}] \text{ holds} \iff h \in (\sim\mathrm{P_A})^* \iff d(h,h) \notin \mathrm{P_A} \iff$$
$$\iff H[\overline{h}] \text{ is not provable}$$
- as PA is correct $H[\overline{h}]$ cannot be provable
otherwise $H[\overline{h}]$ would be false and provable
- hence $H[\overline{h}]$ is true, but not provable

■

Definition
we consider an axiom system $\mathcal{S}$ over the language of PA such that
1. $\mathcal{S}$ includes axioms for first-order with equality
2. $\mathcal{S}$ has rules *modus ponens* and generalisation
3. in addition $\mathcal{S}$ has arbitrary non-logical axioms

Definition
- $\mathcal{S}$ is consistent if $\neg(\mathcal{S} \vdash F$ and $\mathcal{S} \vdash \neg F)$
- $\mathcal{S}$ is ω-inconsistent if
  $\mathcal{S} \vdash \exists w F(w)$ and $\mathcal{S} \vdash \neg F(\overline{0}), \ldots, \mathcal{S} \vdash \neg F(\overline{n}), \mathcal{S} \vdash \neg F(\overline{n+1}), \ldots$
- $\mathcal{S}$ is ω-consistent if $\neg(\omega$-inconsistent$)$

Fact
ω-consistency implies consistency

Gödel's Original Formulation

Definition
$\mathcal{S}$ is recursively axiomatisable (axiomatisable) if the set of Gödel numbers of theorems in $\mathcal{S}$ is Σ_1

Example
PA is recursively axiomatisable

Definition
given two systems $\mathcal{S}_1$, $\mathcal{S}_2$ we say that $\mathcal{S}_1$ is a subsystem of $\mathcal{S}_2$ ($\mathcal{S}_2$ is an extension) of $\mathcal{S}_1$), if all provable formulas of $\mathcal{S}_1$ are provable in $\mathcal{S}_2$

Theorem
if PA *is ω-consistent, then it is incomplete*

Towards Gödel's Incompleteness Proof

Theorem ①
If $\mathcal{S}$ is any axiomatisable ω-consistent system in which all true Σ_0-sentences are provable, then $\mathcal{S}$ is incomplete

Theorem ②
all true Σ_0-sentences (of PA*) are provable in* PA

Definition
- $F(v_1)$ represents A if for all $n \in \mathbb{N}$: $F(\overline{n})$ is provable $\iff$ $n \in A$
- $F(v_1, \ldots, v_n)$ represents R if for all $(m_1, \ldots, m_n) \in \mathbb{N}^n$:

$$F(\overline{m}_1, \ldots, \overline{m}_n) \text{ is provable} \iff (m_1, \ldots, m_n) \in R$$

  we also say that $F(v_1, \ldots, v_n)$ represents the relation $R(x_1, \ldots, x_n)$

let P denote the set of Gödel numbers of provable formulas in $\mathcal{S}$ and R the set of Gödel numbers of refutable formulas in $\mathcal{S}$

Lemma
for any formula $H(v_1)$ with Gödel number h
1. *$H(\overline{h})$ is provable in $\mathcal{S}$ iff $h \in P^*$*
2. *$H(\overline{h})$ is refutable in $\mathcal{S}$ iff $h \in R^*$*

Theorem
1. *suppose $\mathcal{S}$ is consistent*
2. *the negation of $H(v_1)$ represents P^* in $\mathcal{S}$*
3. *let $h := \ulcorner H(v_1) \urcorner$*

then the sentence $H(\overline{h})$ is neither provable or refutable in $\mathcal{S}$

Proof.
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Corollary

if P^* is representable in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete

Theorem (a dual of the above theorem)

if R^* is representable in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $\mathcal{S}$ is incomplete

Proof.

as above ∎

Definition

a formula $F(v_1, v_2)$ enumerate a set A in $\mathcal{S}$ if $\forall n \in \mathbb{N}$:

1. if $n \in A$, $\exists m \in \mathbb{N}$ such that $\mathcal{S} \vdash F(\overline{n}, \overline{m})$
2. if $n \notin A$, $\forall m \in \mathbb{N}$ we have $\mathcal{S} \vdash \neg F(\overline{n}, \overline{m})$

a set A is enumerable if $\exists$ formula $F(v_1, v_2)$ that enumerates A

ω-consistency Lemma

Definition

a formula $F(v_1, \dots, v_n, v_{n+1})$ enumerate a relation $R(x_1, \dots, x_n)$ in $\mathcal{S}$ if $\forall n \in \mathbb{N}$:

1. if $R(k_1, \dots, k_n)$ holds, $\exists m \in \mathbb{N}$ such that $\mathcal{S} \vdash F(\overline{k_1}, \dots, \overline{k_n}, \overline{m})$
2. if $R(k_1, \dots, k_n)$ does not hold, $\forall m \in \mathbb{N}$ we have $\mathcal{S} \vdash \neg F(\overline{k_1}, \dots, \overline{k_n}, \overline{m})$

a relation $R(x_1, \dots, x_n)$ is enumerable if $\exists$ formula $F(v_1, \dots, v_n, v_{n+1})$ that enumerates $R(x_1, \dots, x_n)$

Lemma (ω-consistency Lemma)

if $\mathcal{S}$ is ω-consistent, and if set A is enumerable by $F(v_1, v_2)$, then A is representable by $\exists v_2 F(v_1, v_2)$ in $\mathcal{S}$

Theorem

if either P^* or R^* is enumerable in ω-consistent $\mathcal{S}$, then $\mathcal{S}$ is incomplete

Theorem

suppose $F(v_1, v_2)$ enumerate P^* in $\mathcal{S}$; let $f := \ulcorner \forall v_2 \neg F(v_1, v_2) \urcorner$ and let $G := \forall v_2 \neg F(\overline{f}, v_2)$, then:

1. *if $\mathcal{S}$ is consistent, then G is not provable in $\mathcal{S}$*
2. *if $\mathcal{S}$ is ω-consistent, then $\mathcal{S}$ is incomplete*

Theorem (a dual of the above theorem)

suppose $F'(v_1, v_2)$ enumerate R^* in $\mathcal{S}$; let $f' := \ulcorner \exists v_2 F'(v_1, v_2) \urcorner$ and let $G' := \exists v_2 F'(\overline{f}, v_2)$, then:

1. *if $\mathcal{S}$ is consistent, then G' is not provable in $\mathcal{S}$*
2. *if $\mathcal{S}$ is ω-consistent, then $\mathcal{S}$ is incomplete*