Gödel's Incompleteness Theorem

Georg Moser

Institute of Computer Science @ UIBK

Winter 2011

Homework

- Chapter IV, Appendix, Exercise 1.

Homework

- Chapter IV, Appendix, Exercise 1.
- Chapter IV, Appendix, Exercise 2.

Homework

- Chapter IV, Appendix, Exercise 1.
- Chapter IV, Appendix, Exercise 2.
- Chapter IV, Appendix, Exercise 3.

Homework

- Chapter IV, Appendix, Exercise 1.
- Chapter IV, Appendix, Exercise 2.
- Chapter IV, Appendix, Exercise 3.
- Chapter IV, Appendix, Exercise 4.

Homework

- Chapter IV, Appendix, Exercise 1.
- Chapter IV, Appendix, Exercise 2.
- Chapter IV, Appendix, Exercise 3.
- Chapter IV, Appendix, Exercise 4.
- Chapter IV, Appendix, Exercise 5.

Homework

- Chapter IV, Appendix, Exercise 1.
- Chapter IV, Appendix, Exercise 2.
- Chapter IV, Appendix, Exercise 3.
- Chapter IV, Appendix, Exercise 4.
- Chapter IV, Appendix, Exercise 5.
- Chapter IV, Appendix, Exercise 6.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Proof of Theorem ①

Recall Theorem ①

If $\mathcal{S}$ is any axiomatisable ω-consistent system in which all true Σ_0-sentences are provable, then $\mathcal{S}$ is incomplete

Proof of Theorem ①

Recall Theorem ①

If $\mathcal{S}$ is any axiomatisable ω-consistent system in which all true Σ_0-sentences are provable, then $\mathcal{S}$ is incomplete

Theorem (almost Theorem ①)

if $\mathcal{S}$ is an axiomatisable ω-consistent system such that all Σ_1-sets are enumerable, then $\mathcal{S}$ is incomplete

Proof of Theorem ①

Recall Theorem ①

If $\mathcal{S}$ is any axiomatisable ω-consistent system in which all true Σ_0-sentences are provable, then $\mathcal{S}$ is incomplete

Theorem (almost Theorem ①)

if $\mathcal{S}$ is an axiomatisable ω-consistent system such that all Σ_1-sets are enumerable, then $\mathcal{S}$ is incomplete

Lemma

if all true Σ_0-sentences are provable in $\mathcal{S}$, then all Σ_1-sets are enumerable in $\mathcal{S}$

Proof.

on the whiteboard ∎

Theorem

if $\mathcal{S}$ is any axiomatisable ω-consistent system in which no false Σ_0-sentence is provable, then $\mathcal{S}$ is incomplete

Proof.

Theorem

if $\mathcal{S}$ is any axiomatisable ω-consistent system in which no false Σ_0-sentence is provable, then $\mathcal{S}$ is incomplete

Proof.

- theorem follows as a corollary of Theorem ①

Theorem

if $\mathcal{S}$ is any axiomatisable ω-consistent system in which no false Σ_0-sentence is provable, then $\mathcal{S}$ is incomplete

Proof.

- theorem follows as a corollary of Theorem ①
- on the other hand the theorem can be proven directly, then Theorem ① follows as corollary

∎

Theorem

if $\mathcal{S}$ is any axiomatisable ω-consistent system in which no false Σ_0-sentence is provable, then $\mathcal{S}$ is incomplete

Proof.

- theorem follows as a corollary of Theorem ①
- on the other hand the theorem can be proven directly, then Theorem ① follows as corollary ∎

Recall Theorem ②

all true Σ_0-sentences (of PA*) are provable in* PA

Theorem

if $\mathcal{S}$ is any axiomatisable ω-consistent system in which no false Σ_0-sentence is provable, then $\mathcal{S}$ is incomplete

Proof.

- theorem follows as a corollary of Theorem ①
- on the other hand the theorem can be proven directly, then Theorem ① follows as corollary ∎

Recall Theorem ②

all true Σ_0-sentences (of PA*) are provable in* PA

Definition

a system $\mathcal{S}$ is Σ_0-complete if all true Σ_0-sentences are provable in $\mathcal{S}$

Σ_0-Completeness

Definition

a Σ_0-sentence is correctly decidable in $\mathcal{S}$, if it is either true and provable or false and refutable in $\mathcal{S}$

Σ_0-Completeness

Definition

a Σ_0-sentence is correctly decidable in $\mathcal{S}$, if it is either true and provable or false and refutable in $\mathcal{S}$

Lemma

together the following two conditions are sufficient for $\mathcal{S}$ to be Σ_0-complete:

- C_1 $\forall$ *atomic Σ_0-sentence A, A is correctly decidable*
- C_2 $\forall$ *Σ_0-formula $F(v_1)$, $\forall$ $n \in \mathbb{N}$: if*

$$\mathcal{S} \vdash F(\overline{0}), \ldots, \mathcal{S} \vdash F(\overline{n})$$

then $\mathcal{S} \vdash (\forall v_1 \leqslant \overline{n}) F(v_1)$

Proof.

on the whiteboard ∎

Lemma

together the following three conditions are sufficient for $\mathcal{S}$ to be Σ_0-complete:

- D_1 $\forall$ *true atomic Σ_0-sentence A, A is provable*
- D_2 $\forall m, n$: $m \neq n$: $\mathcal{S} \vdash \overline{m} \neq \overline{n}$
- D_3 $\forall$ *variable w*, $\forall n \in \mathbb{N}$:

$$\mathcal{S} \vdash w \leqslant \overline{n} \rightarrow (w = \overline{0} \vee \cdots \vee w = \overline{n})$$

more precisely: $D_1 - D_3$ imply C_1 and D_3 implies C_2

Proof.

on the whiteboard ∎

A variant of Robinson's Q

Definition (Q)

$$
\begin{array}{ll}
N_1: & v_1' = v_2' \to v_1 = v_2 \\
N_2: & \overline{0} \neq v_1' \\
N_3: & (v_1 + \overline{0}) = v_1 \\
N_4: & (v_1 + v_2') = (v_1 + v_2)' \\
N_5: & (v_1 \cdot \overline{0}) = \overline{0} \\
N_6: & (v_1 \cdot v_2') = ((v_1 \cdot v_2) + v_1) \\
N_7: & (v_1 \leqslant \overline{0}) \leftrightarrow (v_1 = \overline{0}) \\
N_8: & (v_1 \leqslant v_2') \leftrightarrow (v_1 \leqslant v_2 \vee v_1 = v_2') \\
N_9: & (v_1 \leqslant v_2) \vee (v_2 \leqslant v_1)
\end{array}
$$

A variant of Robinson's Q

Definition (Q)

$$
\begin{aligned}
N_1: &\quad v_1' = v_2' \to v_1 = v_2 \\
N_2: &\quad \overline{0} \neq v_1' \\
N_3: &\quad (v_1 + \overline{0}) = v_1 \\
N_4: &\quad (v_1 + v_2') = (v_1 + v_2)' \\
N_5: &\quad (v_1 \cdot \overline{0}) = \overline{0} \\
N_6: &\quad (v_1 \cdot v_2') = ((v_1 \cdot v_2) + v_1) \\
N_7: &\quad (v_1 \leqslant \overline{0}) \leftrightarrow (v_1 = \overline{0}) \\
N_8: &\quad (v_1 \leqslant v_2') \leftrightarrow (v_1 \leqslant v_2 \vee v_1 = v_2') \\
N_9: &\quad (v_1 \leqslant v_2) \vee (v_2 \leqslant v_1)
\end{aligned}
$$

let Q_0 be Q without the axiom N_9

Definition (R)

$$\begin{aligned}
\Omega_1: &\quad \overline{m} + \overline{n} = \overline{k} \quad \text{if } m + n = k \\
\Omega_2: &\quad \overline{m} \cdot \overline{n} = \overline{k} \quad \text{if } m \cdot n = k \\
\Omega_3: &\quad \overline{m} \neq \overline{n} \quad \text{if } m \neq n \\
\Omega_4: &\quad v_1 \leqslant \overline{n} \leftrightarrow (w = \overline{0} \vee \cdots \vee w = \overline{n}) \\
\Omega_5: &\quad v_1 \leqslant \overline{n} \vee \overline{n} \leqslant v_1
\end{aligned}$$

Definition (R)

$$\Omega_1: \quad \overline{m} + \overline{n} = \overline{k} \quad \text{if } m + n = k$$

$$\Omega_2: \quad \overline{m} \cdot \overline{n} = \overline{k} \quad \text{if } m \cdot n = k$$

$$\Omega_3: \quad \overline{m} \neq \overline{n} \quad \text{if } m \neq n$$

$$\Omega_4: \quad v_1 \leqslant \overline{n} \leftrightarrow (w = \overline{0} \vee \cdots \vee w = \overline{n})$$

$$\Omega_5: \quad v_1 \leqslant \overline{n} \vee \overline{n} \leqslant v_1$$

let R_0 be R without the schema Ω_5