Gödel's Incompleteness Theorem

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Homework

- Chapter IV, Appendix, Exercise 1.
- Chapter IV, Appendix, Exercise 2.
- Chapter IV, Appendix, Exercise 3.
- Chapter IV, Appendix, Exercise 4.
- Chapter IV, Appendix, Exercise 5.
- Chapter IV, Appendix, Exercise 6.

Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of $\mathcal{L}$

Tarski's Theorem for Arithmetic

the language $\mathcal{L}_E$, concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1-relations

Gödel's Proof

ω-consistency, a basic incompleteness theorem, ω-consistency lemma, Σ_0-complete subsystems, ω-incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Proof of Theorem ①

Recall Theorem ①

If $\mathcal{S}$ is any axiomatisable ω-consistent system in which all true Σ_0-sentences are provable, then $\mathcal{S}$ is incomplete

Theorem (almost Theorem ①)

if $\mathcal{S}$ is an axiomatisable ω-consistent system such that all Σ_1-sets are enumerable, then $\mathcal{S}$ is incomplete

Lemma

if all true Σ_0-sentences are provable in $\mathcal{S}$, then all Σ_1-sets are enumerable in $\mathcal{S}$

Proof.

on the whiteboard ∎

Theorem
if $\mathcal{S}$ is any axiomatisable ω-consistent system in which no false Σ_0-sentence is provable, then $\mathcal{S}$ is incomplete

Proof.
- theorem follows as a corollary of Theorem ①
- on the other hand the theorem can be proven directly, then Theorem ① follows as corollary ∎

Recall Theorem ②
all true Σ_0-sentences (of PA) are provable in PA

Definition
a system $\mathcal{S}$ is Σ_0-complete if all true Σ_0-sentences are provable in $\mathcal{S}$

Σ_0-Completeness

Definition
a Σ_0-sentence is correctly decidable in $\mathcal{S}$, if it is either true and provable or false and refutable in $\mathcal{S}$

Lemma
together the following two conditions are sufficient for $\mathcal{S}$ to be Σ_0-complete:

C_1 $\forall$ *atomic Σ_0-sentence A, A is correctly decidable*

C_2 $\forall$ *Σ_0-formula $F(v_1)$, $\forall$ $n \in \mathbb{N}$: if*

$$\mathcal{S} \vdash F(\overline{0}), \ldots, \mathcal{S} \vdash F(\overline{n})$$

then $\mathcal{S} \vdash (\forall v_1 \leqslant \overline{n}) F(v_1)$

Proof.
on the whiteboard ∎

Lemma
together the following three conditions are sufficient for $\mathcal{S}$ to be Σ_0-complete:

D_1 $\forall$ *true atomic Σ_0-sentence A, A is provable*

D_2 $\forall m, n$: $m \neq n$: $\mathcal{S} \vdash \overline{m} \neq \overline{n}$

D_3 $\forall$ *variable w, $\forall n \in \mathbb{N}$:*

$$\mathcal{S} \vdash w \leqslant \overline{n} \rightarrow (w = \overline{0} \vee \cdots \vee w = \overline{n})$$

more precisely: $D_1 - D_3$ imply C_1 and D_3 implies C_2

Proof.
on the whiteboard ∎

A variant of Robinson's Q

Definition (Q)

$$\begin{aligned}
N_1: &\quad v_1' = v_2' \rightarrow v_1 = v_2 \\
N_2: &\quad \overline{0} \neq v_1' \\
N_3: &\quad (v_1 + \overline{0}) = v_1 \\
N_4: &\quad (v_1 + v_2') = (v_1 + v_2)' \\
N_5: &\quad (v_1 \cdot \overline{0}) = \overline{0} \\
N_6: &\quad (v_1 \cdot v_2') = ((v_1 \cdot v_2) + v_1) \\
N_7: &\quad (v_1 \leqslant \overline{0}) \leftrightarrow (v_1 = \overline{0}) \\
N_8: &\quad (v_1 \leqslant v_2') \leftrightarrow (v_1 \leqslant v_2 \vee v_1 = v_2') \\
N_9: &\quad (v_1 \leqslant v_2) \vee (v_2 \leqslant v_1)
\end{aligned}$$

let Q_0 be Q without the axiom N_9

Definition (R)

$$\Omega_1: \quad \overline{m} + \overline{n} = \overline{k} \quad \text{if } m + n = k$$

$$\Omega_2: \quad \overline{m} \cdot \overline{n} = \overline{k} \quad \text{if } m \cdot n = k$$

$$\Omega_3: \quad \overline{m} \neq \overline{n} \quad \text{if } m \neq n$$

$$\Omega_4: \quad v_1 \leqslant \overline{n} \leftrightarrow (w = \overline{0} \vee \cdots \vee w = \overline{n})$$

$$\Omega_5: \quad v_1 \leqslant \overline{n} \vee \overline{n} \leqslant v_1$$

let R_0 be R without the schema Ω_5