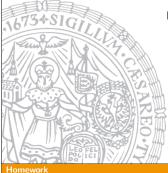


Gödel's Incompleteness Theorem

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Outline of the Lecture

General Idea Behind Gödel's Proof

abstract forms of Gödel's, Tarski's theorems, undecidable sentences of ${\cal L}$

Tarski's Theorem for Arithmetic

the language \mathcal{L}_E , concatenation and Gödel numbering, Tarski's theorem, the axiom system PE, arithmetisation of the axiom system, arithmetic without exponentiation, incompleteness of PA, Σ_1 -relations

Gödel's Proof

 ω -consistency, a basic incompleteness theorem, ω -consistency lemma, Σ_0 complete subsystems, ω -incompleteness of PA

Rosser Systems

abstract incompleteness theorems after Rosser, general separation principle, Rosser's undecidable sentence, Gödel and Rosser sentences compared, more on separation

Homework

- Chapter IV, Appendix, Exercise 1.
- Chapter IV, Appendix, Exercise 2.
- Chapter IV, Appendix, Exercise 3.
- Chapter IV, Appendix, Exercise 4.
- Chapter IV, Appendix, Exercise 5.
- Chapter IV, Appendix, Exercise 6.

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-Consistency Lemma

Proof of Theorem (1)

Recall Theorem ①

If S is any axiomatisable ω -consistent system in which all true Σ_0 -sentences are provable, then S is incomplete

Theorem (almost Theorem ①)

if S is an axiomatisable ω -consistent system such that all Σ_1 -sets are enumerable, then S is incomplete

Lemma

if all true Σ_0 -sentences are provable in S, then all Σ_1 -sets are enumerable in S

Proof.

on the whiteboard



Theorem

if S is any axiomatisable ω -consistent system in which no false Σ_0 -sentence is provable, then S is incomplete

Proof.

- theorem follows as a corollary of Theorem ①
- on the other hand the theorem can be proven directly, then Theorem ① follows as corollary

Recall Theorem ②

all true Σ_0 -sentences (of PA) are provable in PA

Definition

a system S is Σ_0 -complete if all true Σ_0 -sentences are provable in S

Σ_0 -Completeness

Definition

a Σ_0 -sentence is correctly decidable in S, if it is either true and provable or false and refutable in ${\cal S}$

Lemma

together the following two conditions are sufficient for S to be Σ_0 -complete:

 $C_1 \forall atomic \Sigma_0$ -sentence A, A is correctly decidable

 $C_2 \ \forall \ \Sigma_0$ -formula $F(v_1), \ \forall \ n \in \mathbb{N}$: if

$$S \vdash F(\overline{0}), \ldots, S \vdash F(\overline{n})$$

then $S \vdash (\forall v_1 \leqslant \overline{n}) F(v_1)$

Proof.

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Lemma

together the following three conditions are sufficient for S to be Σ_0 -complete:

 $D_1 \forall true atomic \Sigma_0$ -sentence A, A is provable

 $D_2 \ \forall m, n: m \neq n: S \vdash \overline{m} \neq \overline{n}$

 $D_3 \ \forall \ variable \ w, \ \forall n \in \mathbb{N}$:

$$S \vdash w \leqslant \overline{n} \rightarrow (w = \overline{0} \lor \cdots \lor w = \overline{n})$$

more precisely: $D_1 - D_3$ imply C_1 and D_3 implies C_2

Proof.

on the whiteboard

A variant of Robinson's Q

Definition (Q)

$$N_1: v_1' = v_2' \to v_1 = v_2$$

$$N_2$$
: $\overline{0} \neq v_1'$

$$N_3$$
: $(v_1+\overline{0})=v_1$

$$N_4$$
: $(v_1 + v_2') = (v_1 + v_2)'$

$$N_5$$
: $(v_1 \cdot \overline{0}) = \overline{0}$

$$N_6$$
: $(v_1 \cdot v_2') = ((v_1 \cdot v_2) + v_1)$

$$N_7$$
: $(v_1 \leqslant \overline{0}) \leftrightarrow (v_1 = \overline{0})$

$$N_8$$
: $(v_1 \leqslant v_2') \leftrightarrow (v_1 \leqslant v_2 \lor v_1 = v_2')$

$$N_9$$
: $(v_1 \leqslant v_2) \lor (v_2 \leqslant v_1)$

let Q_0 be Q without the axiom N_9

Some Σ_0 -complete Subsystems of PA

Definition (R)

 Ω_1 : $\overline{m} + \overline{n} = \overline{k}$ if m + n = k

 Ω_2 : $\overline{m} \cdot \overline{n} = \overline{k}$ if $m \cdot n = k$

 Ω_3 : $\overline{m} \neq \overline{n}$ if $m \neq n$

 Ω_4 : $v_1 \leqslant \overline{n} \leftrightarrow (w = \overline{0} \lor \cdots \lor w = \overline{n})$

 Ω_5 : $v_1 \leqslant \overline{n} \lor \overline{n} \leqslant v_1$

let R_0 be R without the schema Ω_5

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