Introduction to Model Checking

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Outline

- Notations
- Transition systems
- Model Checking of Linear Time Properties
- Regular Languages
  - Finite Automata
  - Büchi Automata
  - Generalized Büchi Automata

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- states: $S = \{s, t, \dots\}$
- sequences of states: $\rho = s_1 s_2 s_3 \dots \in S^\omega$
- no distinction between set and its characterizing vector
  example: if $AP = \{a_1, a_2, a_3\}$ then $w \in (2^{AP})^\omega$ is sequence of sets or equivalently, sequence of bitvectors

$$w = \{a_1, a_2\} \varnothing \{a_1, a_3\} \dots = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \dots$$

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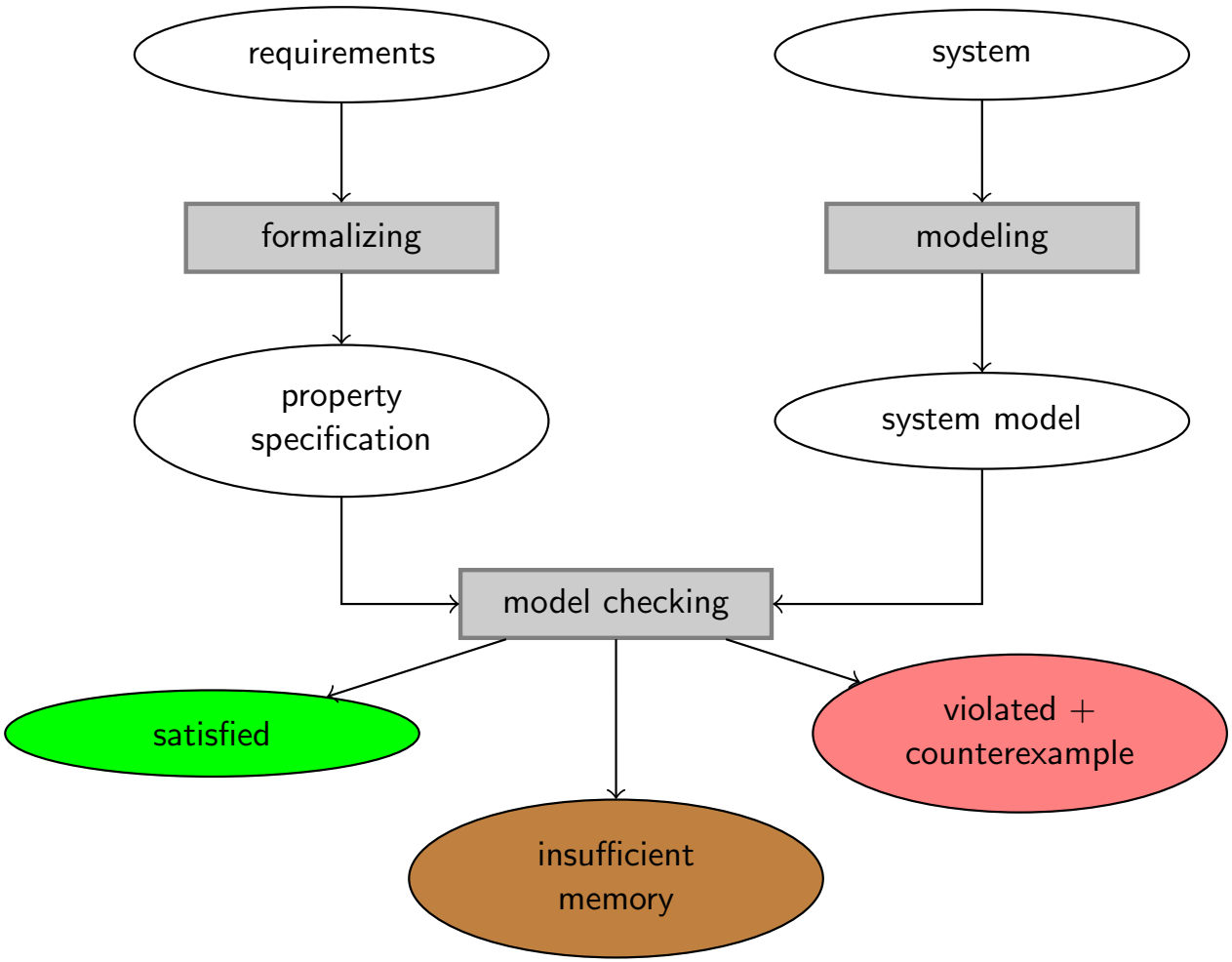

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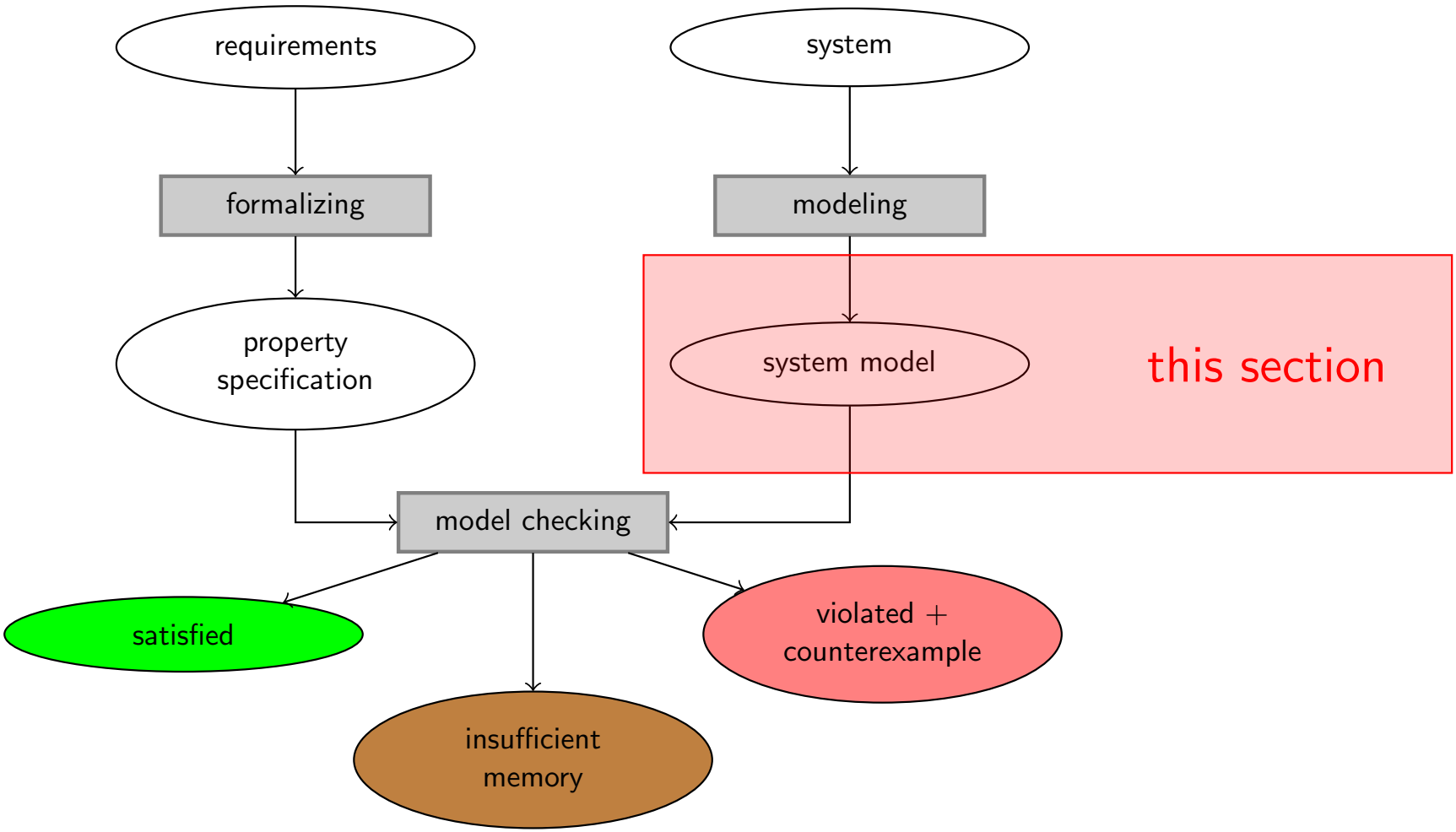

Transition systems

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- digraphs where nodes represent states, and edges model transitions
- state:
  - the current phase of a traffic light
  - the current values of all program variables + the program counter
- transition: ("state change")
  - a switch from one phase to the next one
  - the execution of a program statement

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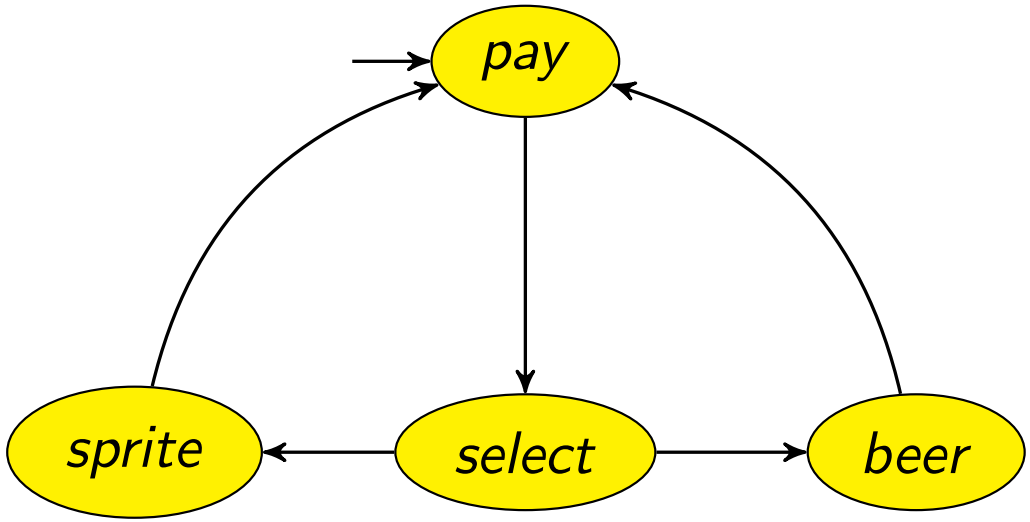
notation: $s \rightarrow s'$ instead of $(s, s') \in \longrightarrow$

A beverage vending machine



states?, transitions?, initial states?

$$TS = (S = \{pay, select, sprite, beer\}, \rightarrow, I = \{pay\}, AP, L)$$

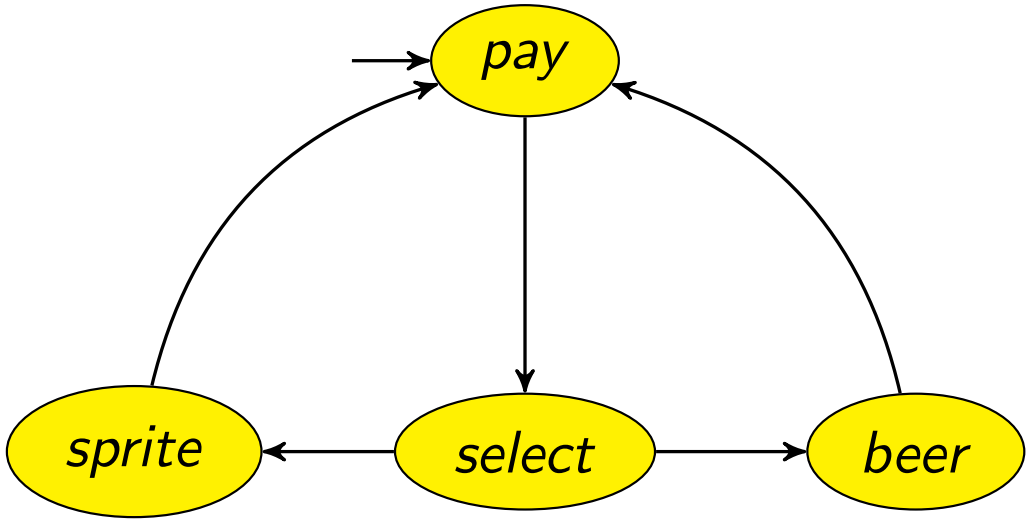
where $\rightarrow$ is the following set of transitions

$$pay \rightarrow select$$

$$select \rightarrow beer \qquad select \rightarrow sprite$$

$$beer \rightarrow pay \qquad sprite \rightarrow pay$$

Atomic propositions?



(paid, beer paid, sprite paid will be displayed)

$$AP = \{\text{paid}, \text{sprite}, \text{beer}\}$$
$$L(pay) = \varnothing$$
$$L(select) = \{\text{paid}\}$$
$$L(beer) = \{\text{beer}, \text{paid}\}$$
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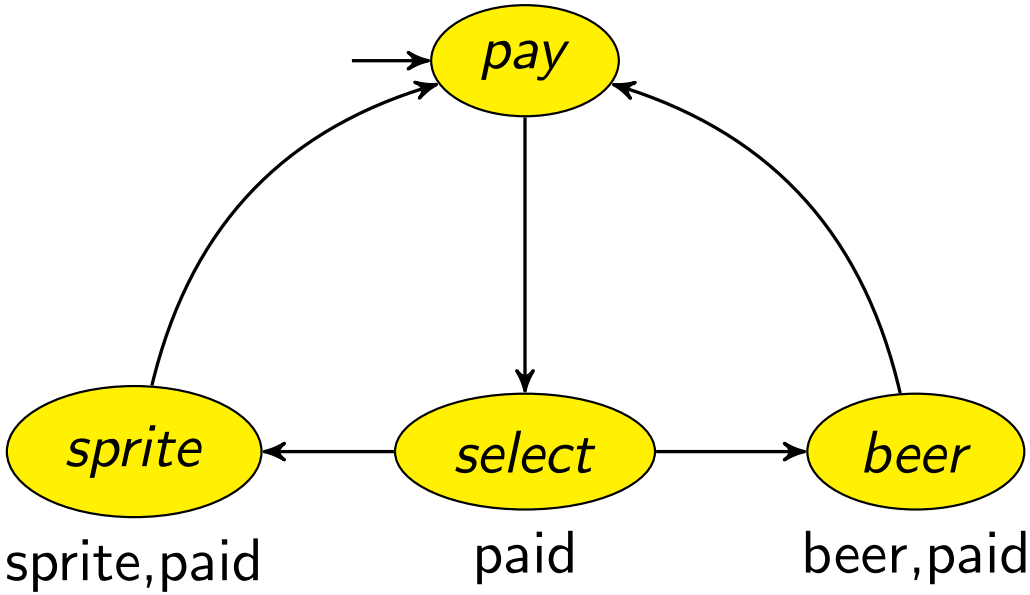

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- to model concurrency by interleaving
  - no assumption about the relative speed of processes
- to model implementation freedom
  - only describes what a system should do, not how
- to model under-specified systems, or abstractions of real systems
  - use incomplete information

in automata theory, nondeterminism may be exponentially more succinct, but that's not the issue here!

Executions

- execution ϱ of TS: sequence of states

$$\varrho \;=\; s_0\, s_1\, \ldots\, s_n\, \ldots$$

such that

- $s_i \rightarrow s_{i+1}$ for all $0 \leqslant i \in \mathbb{N}$
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- trace of an execution: sequence of sets of atomic propositions, i.e., $\mathit{trace}(\varrho) \in (2^{AP})^{\omega}$

$$\mathit{trace}(\varrho) = L(s_0)\; L(s_1)\; L(s_2)\; L(s_3)\; \ldots$$

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- $Traces(TS)$: set of all traces of all executions of TS
  it defines the observable behaviour of TS

Example

$$\varrho = \mathit{pay} \to \mathit{select} \to \mathit{sprite} \to \mathit{pay} \to \mathit{select} \to \mathit{beer} \ldots$$

$$\mathit{trace}(\varrho) = \varnothing\ \{\text{paid}\}\ \{\text{paid}, \text{sprite}\}\ \varnothing\ \{\text{paid}\}\ \{\text{paid}, \text{beer}\}\ \ldots$$

interesting properties

- without having paid there is no sprite (true)
- we will see beer infinitely often (false)
- after each beer another coin is inserted (not expressible, needs AP coin, paid may contain credit card)

states $\sim$ implementation

$\Rightarrow$ properties only speak about atomic propositions, never about states!
(we are not interesting in implementation, only result must be correct)

Summary

transition systems $\neq$ finite automata since

- there are no accept states
- set of states may be countably infinite
  (but in this lecture: only finite sets of states)
- may have infinite branching
- non-determinism has a different role

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$\Rightarrow$ transition systems are appropriate for reactive system behavior

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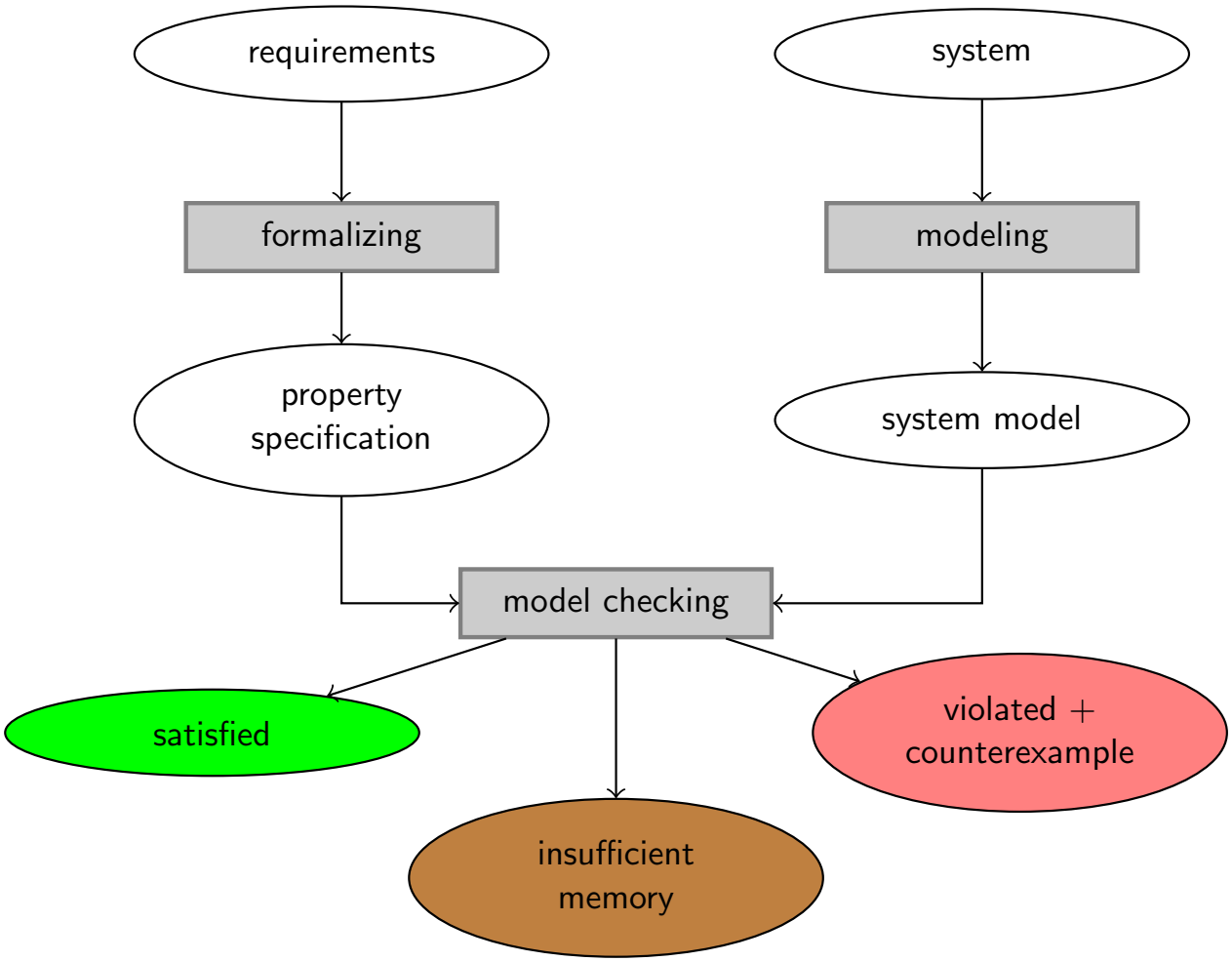

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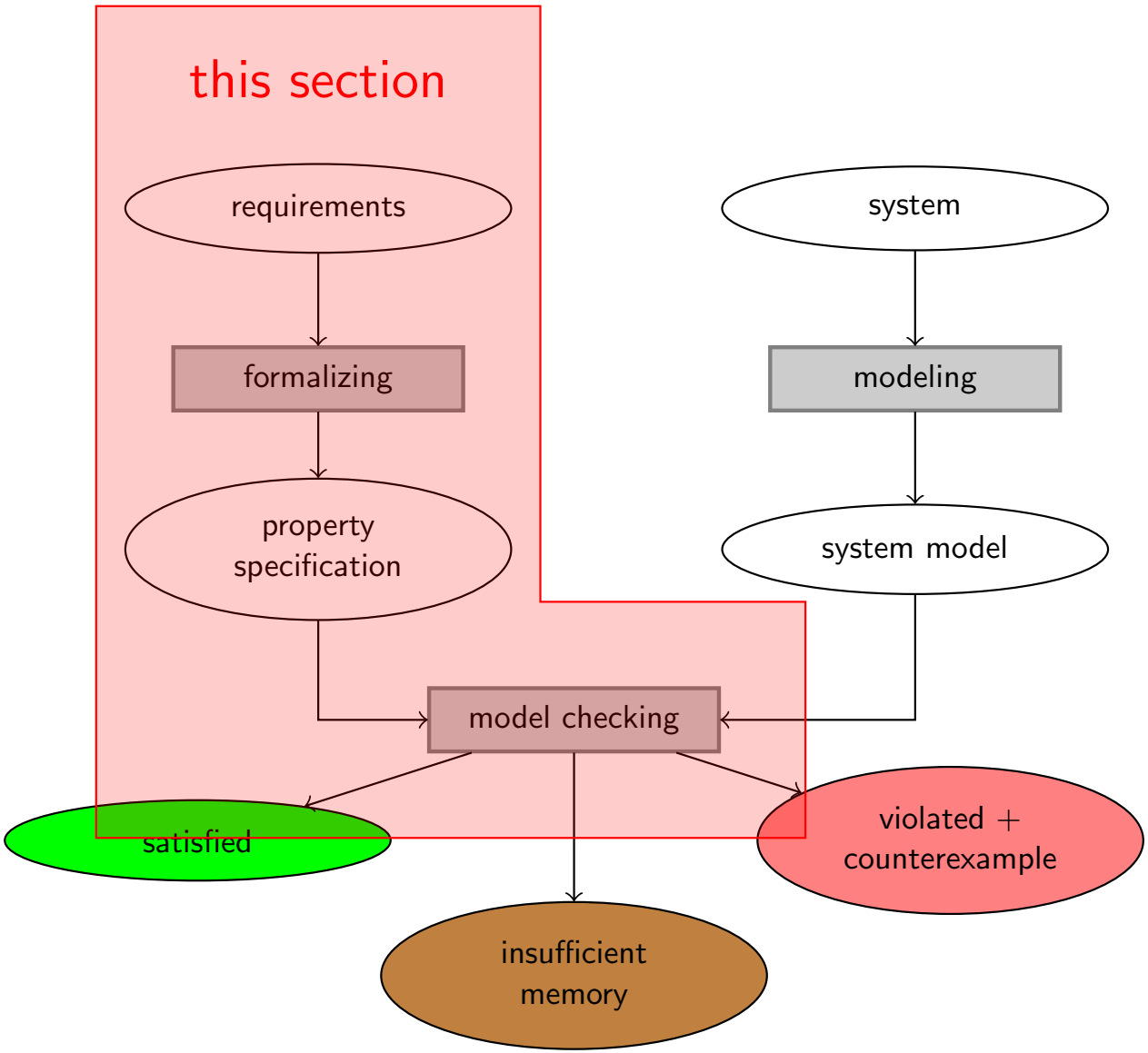

Requirements $\neq$ Specification

requirements

- high-level description (consider scheduler for exclusive access)
  - (the scheduler should be correct)
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$\Rightarrow$ we need requirements in a precise, i.e., mathematical specification

Linear Time Properties

one main idea to specify requirements: describe allowed traces

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- upcoming problems
  - how to specify sets $\mathcal{S}$ conveniently ...
  - ... such that $\mathit{Traces}(TS) \subseteq \mathcal{S}$ can be decided

The requirements of model checking

essentially we need a mechanism to represent the set $\mathcal{S}(R)$ of allowed traces for some requirement R conveniently

- possible classes: finite, regular, context-free, context-sensitive, ...
- model checking requires checking $\mathit{Traces}(\mathit{TS}) \subseteq \mathcal{S}(R)$
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- use regular languages, they are closed under all boolean operations
- possible representations of regular languages
  - regular expressions
  - non-recursive grammars
  - finite automata

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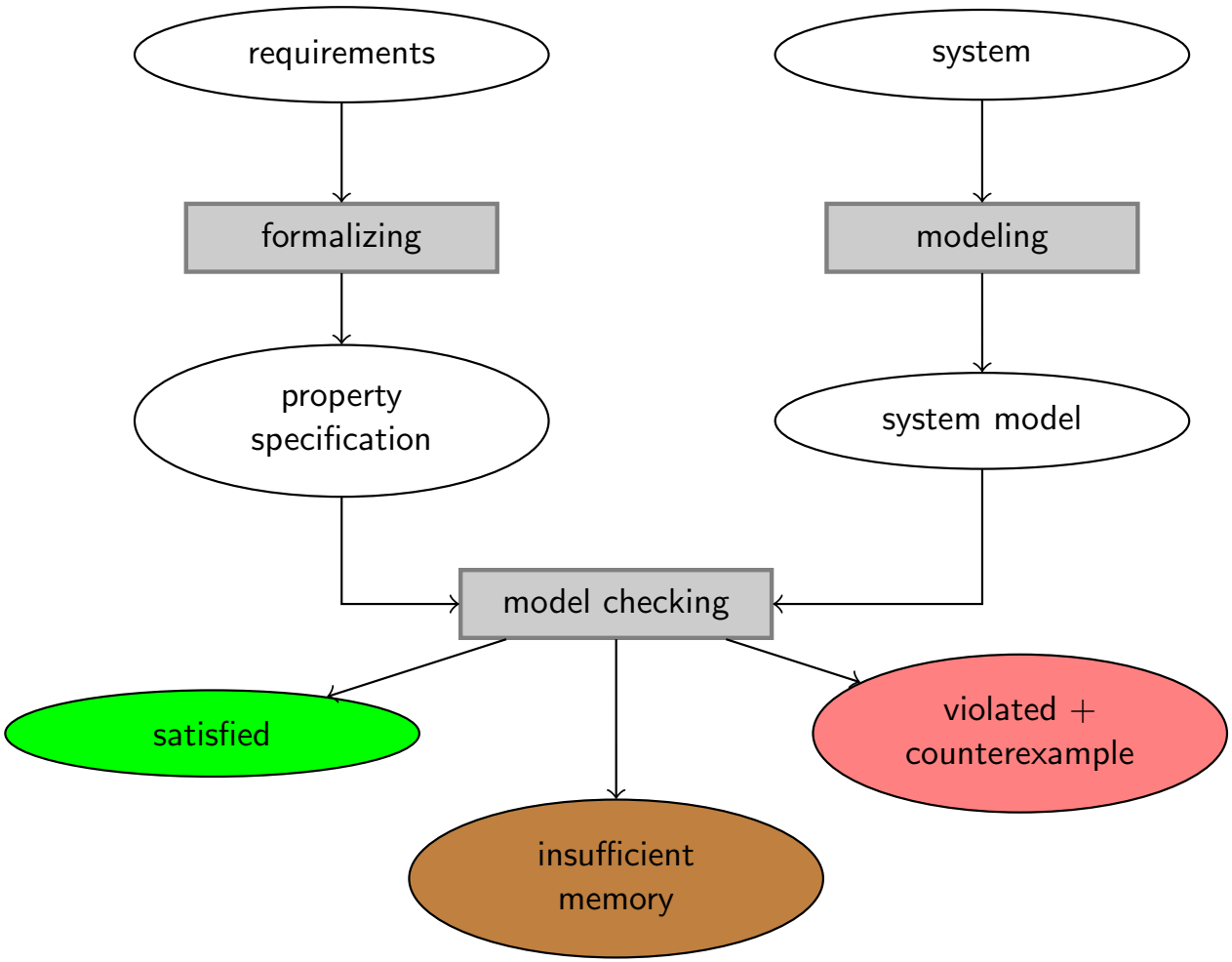

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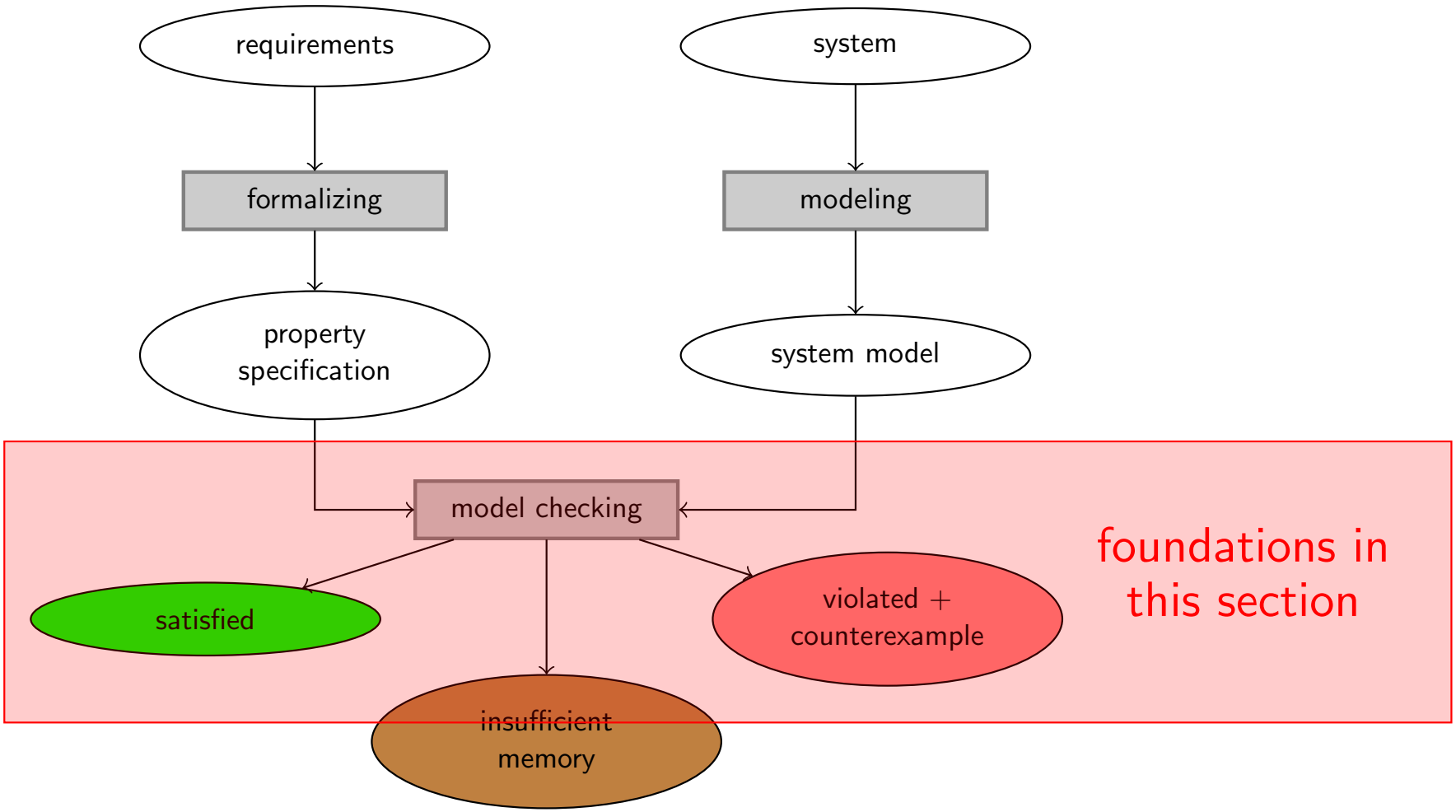

Finite Automata

a nondeterministic finite automaton (NFA) $\mathcal{A}$ is a tuple $(Q, \Sigma, \delta, q_0, F)$ where:

- $Q = \{q_0, \ldots, q_n\}$ is a finite set of states
- Σ is an alphabet
- $\delta : Q \times \Sigma \to 2^Q$ is a transition function
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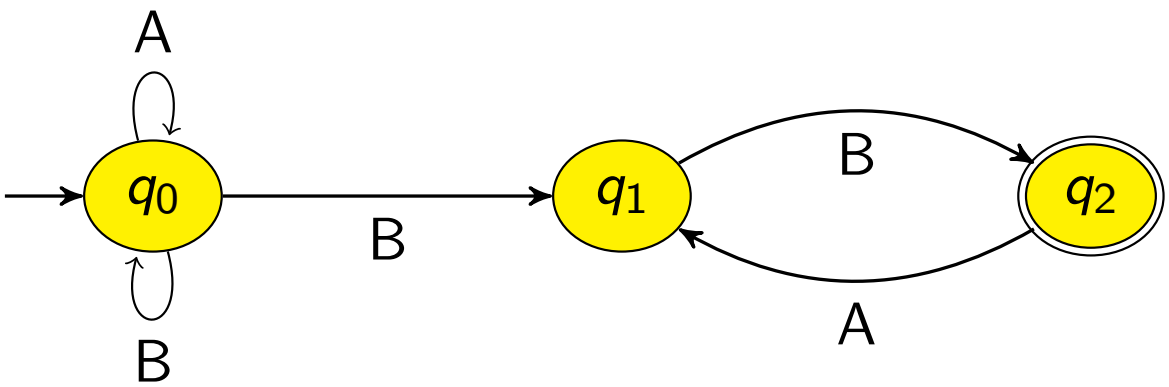
- $F \subseteq Q$ is a set of final (or: accepting) states



later, after definition: $\mathcal{L}(\mathcal{A}) = (A \vee B)^* BB(AB)^*$

Language of an NFA

- NFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ and word $w = A_1 \dots A_n \in \Sigma^*$
- run of w in $\mathcal{A}$: finite sequence $q_0\, q_1\, \dots\, q_n$ such that
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$$\mathcal{L}(\mathcal{A}) = \left\{ w \in \Sigma^* \mid \text{ there exists an accepting run for } w \text{ in } \mathcal{A} \right\}$$

- NFA $\mathcal{A}$ and $\mathcal{A}'$ are equivalent iff $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$
- language $\mathcal{L}$ is regular iff $\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some NFA $\mathcal{A}$

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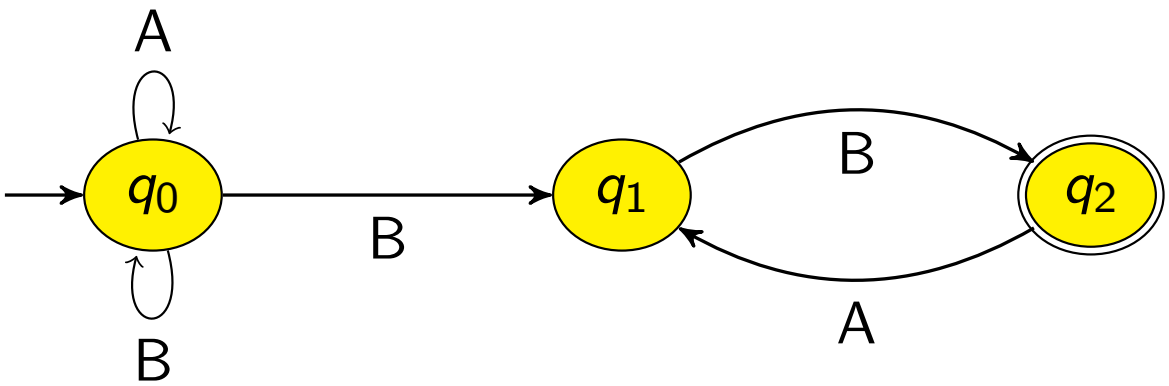

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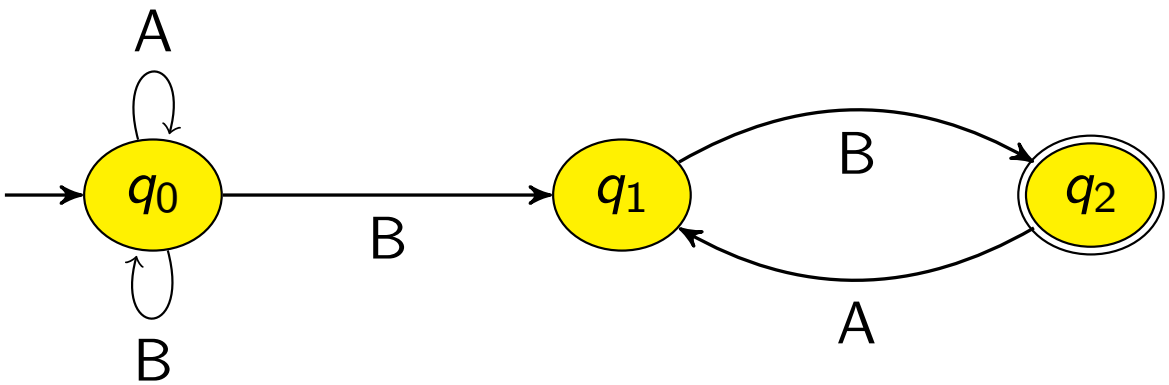

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- NBA $\mathcal{A}$ and $\mathcal{A}'$ are equivalent iff $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$
- language $\mathcal{L}$ is ω-regular iff $\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some NBA $\mathcal{A}$

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- NBAs are GNBAs where $k = 1$
- each GNBA can be translated into equivalent NBA (with states $Q \times \{1, \ldots, \max(1, k)\}$)

GNBAs to specify requirements

Safety properties: (refutation by a finite prefix of an ω-word)

1. always at most one traffic light is showing green
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Liveness properties: (refutation only by whole ω-word)

3. we will see green infinitely often
4. whenever we select sprite then later on we will get a sprite

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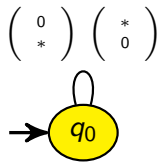
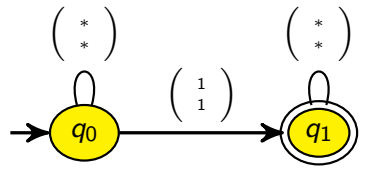
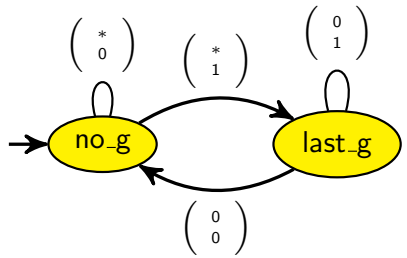
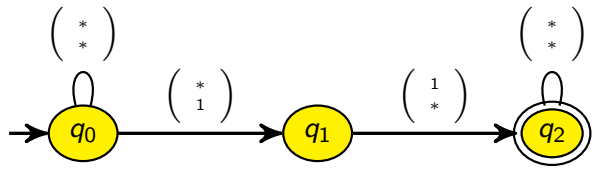
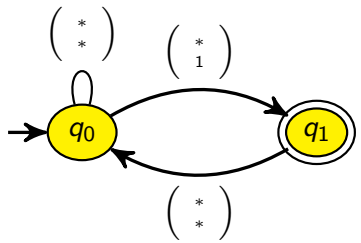
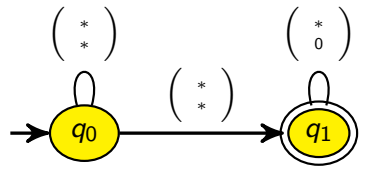
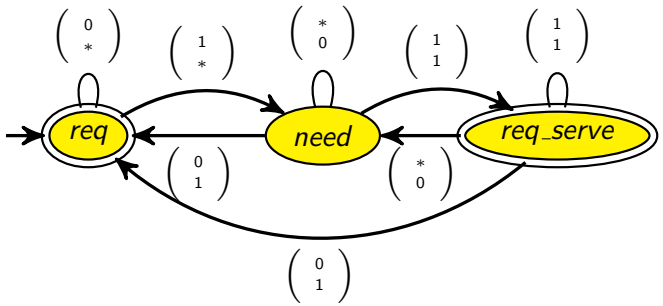
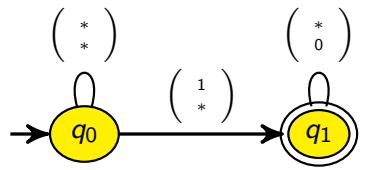
Liveness properties: (refutation only by whole ω-word)

3. we will see green infinitely often
4. whenever we select sprite then later on we will get a sprite

- GNBAs are closed under union, intersection, and negation

$\Rightarrow$ many interesting properties can be expressed by GNBAs

GNBAs to specify requirements

	R	$\neg R$
1.	$\begin{pmatrix} green1 \\ green2 \end{pmatrix}$ $k = 0$ 	
2.	$\begin{pmatrix} red \\ green \end{pmatrix}$ $k = 0$ 	
3.	$\begin{pmatrix} red \\ green \end{pmatrix}$ 	
4.	$\begin{pmatrix} select \\ get \end{pmatrix}$ 	

Recall: requirements of model checking

- model checking requires checking $\mathit{Traces}(\mathit{TS}) \cap \mathcal{S}(\neg R) = \varnothing$
  where $\neg R$ describes forbidden traces

$\Rightarrow$ requirements on class of language

- expressive enough to represent $\mathit{Traces}(\mathit{TS})$
- closure under intersection
- emptyness decidable

Expressing Transition Systems as GNBAs

aim: for $TS = (S, \to, I, AP, L)$ construct $\mathcal{A}_{TS}$ with $\mathcal{L}(\mathcal{A}_{TS}) = \mathit{Traces}(TS)$

problems:

- labels/letters are at the states in TS, but on the transitions in GNBAs
- several initial states in TS, but only one initial state in GNBAs

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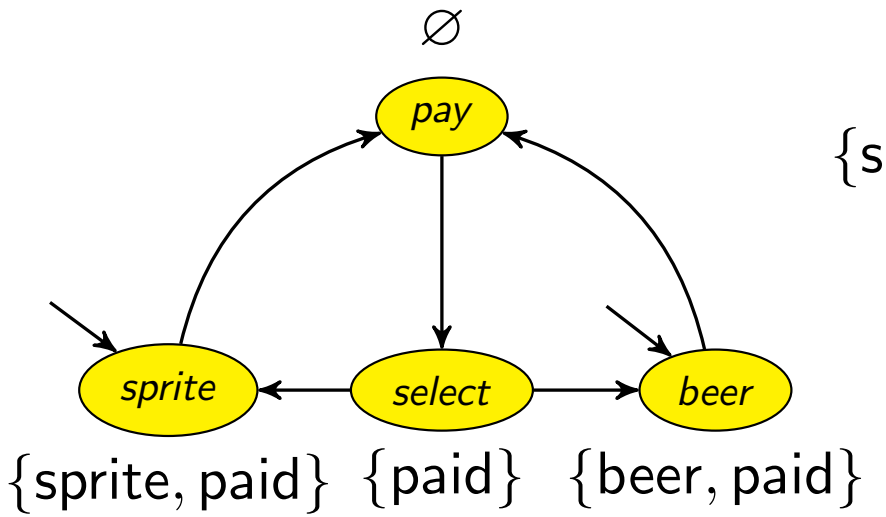
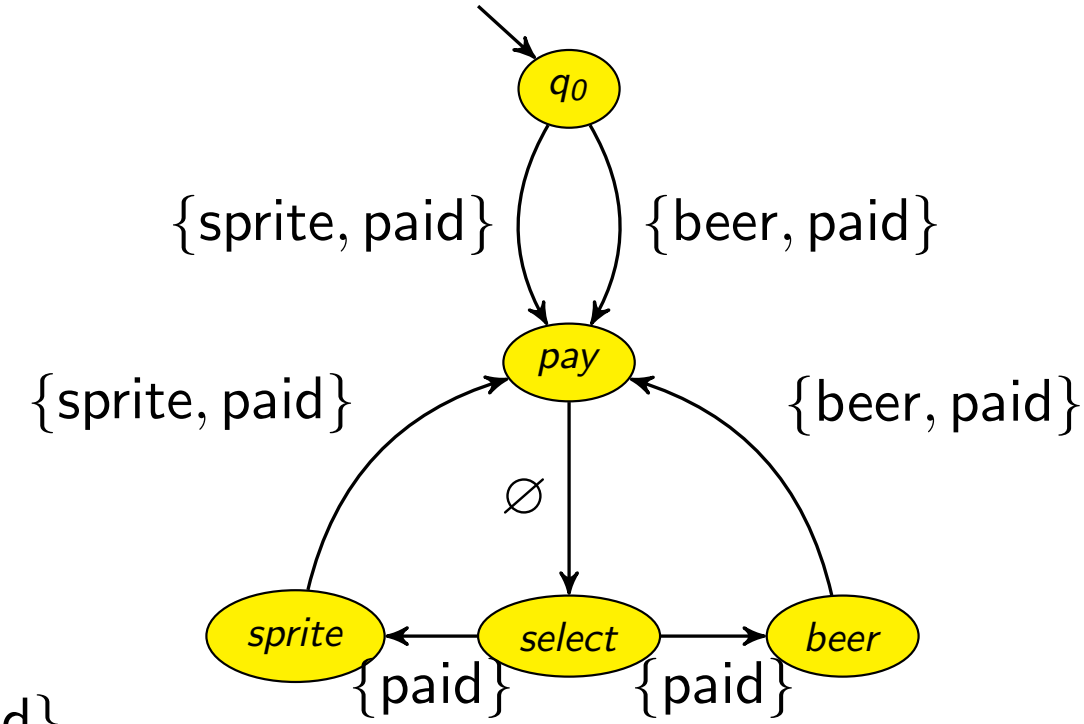
concrete: $\mathcal{A}_{TS} = (S \uplus \{q_0\}, 2^{AP}, \delta, q_0)$ with δ defined as follows

- $\delta(s, A) = \{s' \mid L(s) = A, s \to s'\}$
- $\delta(q_0, A) = \bigcup_{s \in I} \delta(s, A)$

Example



TS



$\mathcal{A}_{TS}$

Soundness of $\mathcal{A}_{TS}$

Theorem

$$\mathcal{L}(\mathcal{A}_{TS}) = \mathit{Traces}(\mathit{TS})$$

let $w = A_0\, A_1\, A_2\, \ldots$

- $w \in \mathit{Traces}(\mathit{TS})$
- iff exist execution $s_0\, s_1\, s_2\, \ldots$ of TS:
  $s_0 \in I$ and for all $i \in \mathbb{N}$:
  - $L(s_i) = A_i$
  - $s_i \to s_{i+1}$
- iff exist (accepting) run $q_0\, s_1\, s_2\, \ldots$ of $\mathcal{A}_{TS}$:
  $s_1 \in \delta(q_0, A_0)$ and for all $i \geqslant 1$:
  - $s_{i+1} \in \delta(s_i, A_i)$
- iff $w \in \mathcal{L}(\mathcal{A}_{TS})$

GNBA for Intersection

aim: for $\mathcal{A}_i = (\mathcal{Q}_i, \Sigma, \delta_i, q_{0,i}, F_{1,i}, \ldots, F_{k_i,i})$ construct $\mathcal{A}_{\mathcal{A}_1 \cap \mathcal{A}_2}$ with $\mathcal{L}(\mathcal{A}_{\mathcal{A}_1 \cap \mathcal{A}_2}) = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$

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idea: simulate runs in $\mathcal{A}_1$ and $\mathcal{A}_2$ in parallel

- use cartesian product of states
- demand that all final states are visited infinitely often

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concrete: $\mathcal{A}_{\mathcal{A}_1 \cap \mathcal{A}_2} = (Q, \Sigma, \delta, q_0, F'_1, \ldots, F'_{k_1}, F''_1, \ldots, F''_{k_2})$ where

- $Q = Q_1 \times Q_2$

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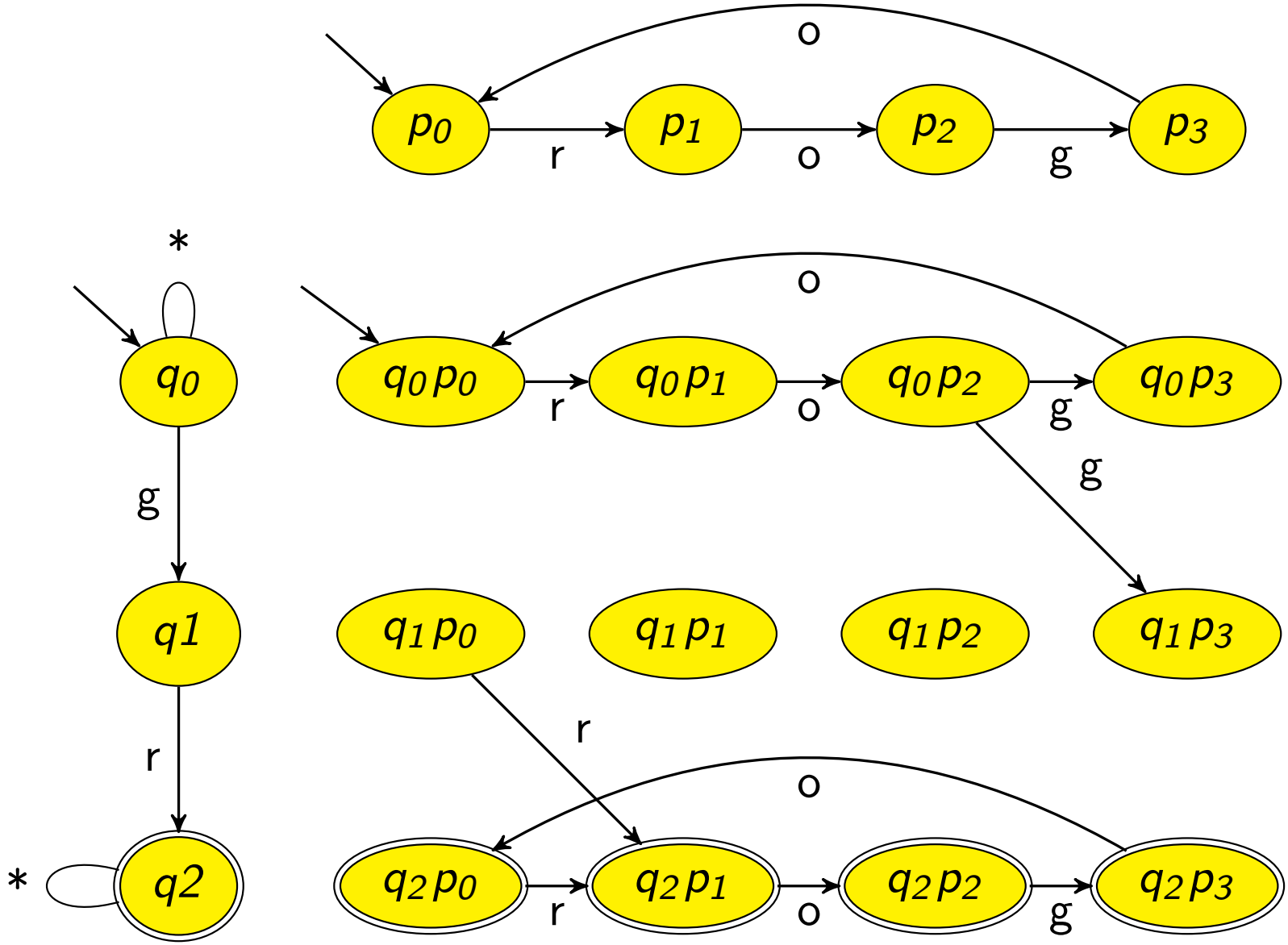
- $F'_j = F_{j,1} \times Q_2$ and $F''_j = Q_1 \times F_{j,2}$

Example: no direct switch from green to red



Soundness of $\mathcal{A}_{\mathcal{A}_1 \cap \mathcal{A}_2}$

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$$\mathcal{L}(\mathcal{A}_{\mathcal{A}_1 \cap \mathcal{A}_2}) = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$$

let $w = A_0\, A_1\, A_2\, \ldots$

- $w \in \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$
- iff exists accepting run $q_{0,1}\, q_{1,1}\, q_{2,1}\, \ldots$ of w in $\mathcal{A}_1$
  and exists accepting run $q_{0,2}\, q_{1,2}\, q_{2,2}\, \ldots$ of w in $\mathcal{A}_2$:
  - $q_{j+1,i} \in \delta_i(q_{j,i}, A_j)$ for all $0 \leqslant j$ and $1 \leqslant i \leqslant 2$
  - for each $1 \leqslant j \leqslant k_i$, $1 \leqslant i \leqslant 2$ there are infinitely many $q_{m,i} \in F_{j,i}$
- iff exist accepting run $(q_{0,1}, q_{0,2})\,(q_{1,1}, q_{1,2})\,(q_{2,1}, q_{2,2})\, \ldots$ of $\mathcal{A}_{\mathcal{A}_1 \cap \mathcal{A}_2}$:
  - $(q_{j+1,1}, q_{j+1,2}) \in \delta((q_{j,1}, q_{j,2}), A_j)$ for all $0 \leqslant j$
  - for each F'_j there are infinitely many $(q_{m,1}, q) \in F_{j,1} \times \mathcal{Q}_2$ and
    for each F''_j there are infinitely many $(q, q_{m,2}) \in \mathcal{Q}_1 \times F_{j,2}$
- iff $w \in \mathcal{L}(\mathcal{A}_{\mathcal{A}_1 \cap \mathcal{A}_2})$

Checking Emptiness of GNBAs

let G be a graph (V, E) with nodes V and edges E

- set $C \subseteq V$ is a cycle of G iff
  for all $v_1, v_2 \in C$ there is a non-empty path from v_1 to v_2

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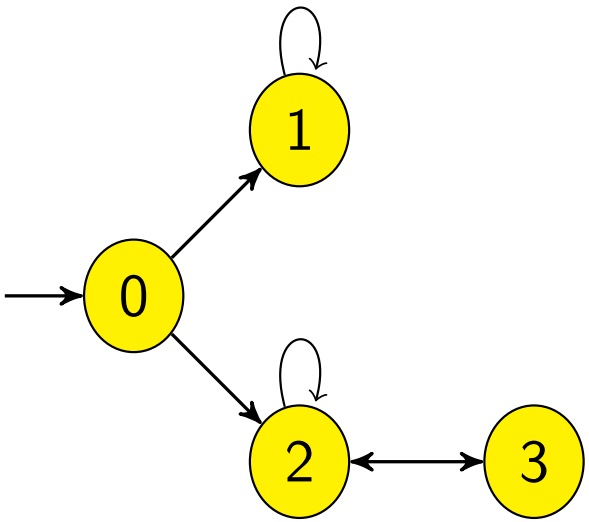
  (C is SCC iff both C is cycle and $C' \supset C$ implies C' is not a cycle)
- remarks:
  - two SCCs C_1 and C_2 are either disjoint or identical
  - the set of SCCs of a graph can be determined in linear time (Kosaraju)



Algorithm for Checking $\mathcal{L}(\mathcal{A}) = \varnothing$ for GNBA $\mathcal{A}$

$\mathcal{A} = (Q, \Sigma, \delta, q_0, F_1, \ldots, F_k)$ accepts at least one ω-word

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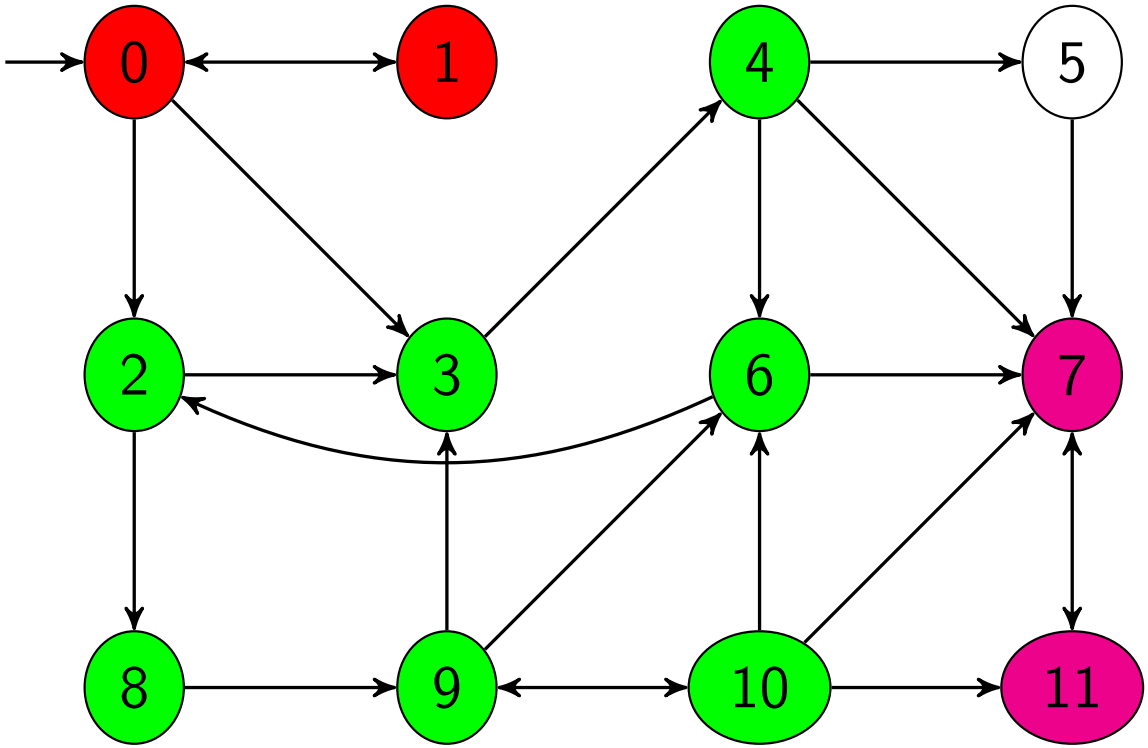
- if no SCC with final states from each F_i reachable from q_0: $\mathcal{L}(\mathcal{A}) = \varnothing$
- otherwise, obtain path from q_0 to $q_1 \in SCC$ and non-empty path from q_1 to q_1 traversing all nodes in the SCC with corresponding words w_{01} and w_{11}
  $\Rightarrow w_{01} w_{11}^{\omega} \in \mathcal{L}(\mathcal{A})$

Example (GNBA where input letters are omitted)



$F_1 = \{0, 9\}$ $F_2 = \{0, 3, 10\}$ $F_3 = \{5, 10, 11\}$

extracted accepted run: $0\,(2\,3\,4\,6\,2\,8\,9\,10\,6)^{\omega}$

Summary

- model checking for linear time properties:
  - specify allowed traces $\mathcal{S}(R)$ or forbidden traces $\mathcal{S}(\neg R)$
  - decide

$$Traces(T) \subseteq \mathcal{S}(R) \qquad \text{or} \qquad Traces(TS) \cap \mathcal{S}(\neg R) = \varnothing$$

- NFA over finite words $\rightarrow$ (G)NBA over infinite words
  (regular languages $\rightarrow$ regular ω-languages)
- GNBAs can encode several requirements (allowed / forbidden traces)
- GNBAs can encode transition systems
- GNBAs are closed under Boolean operations (here: only intersection)
- emptyness of GNBAs is decidable