1. Consider the following propositional formulas

$$\begin{array}{ll} A & (\neg \mathsf{p} \wedge \mathsf{q}) \vee (\mathsf{p} \wedge \neg \mathsf{q}) \\ B & (\mathsf{p} \vee \mathsf{q}) \wedge (\neg \mathsf{p} \vee \neg \mathsf{q}) \\ C & (\mathsf{p} \vee \mathsf{q}) \wedge \neg \mathsf{p} \wedge (\mathsf{q} \to r) \to (\mathsf{q} \wedge r) \end{array}$$

   a) Show that formula A is neither a tautology, nor contradictory, by providing suiable assignments. (2 pts)

   b) Show that formula B is neither a tautology, nor contradictory, by providing suiable assignments. (2 pts)

   c) Show that formula C is valid, either by semantic argumentations or by providing a formal natural deduction or resolution proof. (4 pts)

2. Consider propositional logic and the following attempt at defining resolution, where we assume the definition of *resolution* and *factoring* from the lecture.

   – $\mathsf{Res}_1(\mathcal{C}) := \{D \mid D$ is conclusion of a resolution or factoring with premises in $\mathcal{C}\}$,

   – $\mathsf{Res}_1^0(\mathcal{C}) := \mathcal{C}$, $\mathsf{Res}_1^{n+1}(\mathcal{C}) := \mathsf{Res}_1(\mathsf{Res}_1^n(\mathcal{C}))$, and $\mathsf{Res}_1^*(\mathcal{C}) := \bigcup_{n \geqslant 0} \mathsf{Res}_1^n(\mathcal{C})$.

   a) Give the definition of the operator Res^* as defined in the lecture. (3 pts)

   b) Is the definition of Res_1^* correct, i.e., do we have $\Box \in \mathsf{Res}^*(\mathcal{C})$ iff $\Box \in \mathsf{Res}_1^*(\mathcal{C})$ for any clause set $\mathcal{C}$? Explain your answer. (4 pts)

   c) Consider the orignal definition of the resolution operator, but remove *factoring*. Call the resulting resolution operator Res_2. Is the following statement correct: $\Box \in \mathsf{Res}^*(\mathcal{C})$ iff $\Box \in \mathsf{Res}_2^*(\mathcal{C})$ for any clause set $\mathcal{C}$? Explain your answer. (4 pts)

3. Consider the following sentences:

   ① Some birds can fly and have large mothers.

   ② A bird can fly if all its relatives can fly.

   ③ The relative of a relative, is a relative.

   ④ It is not the case that not all birds do not eat worms.

   ⑤ There exists a large bird that eats fish.

   a) For each of the sentences above, give a first-order formula that formalises the sentence. Use therefore *at most* the following constants, functions and predicates:

      – Individual constants: fish, worms.

- Function constants: father, mother, which are unary.
- Predicates constants: Bird, Small, Medium, Large, Fly, which are unary and $\mathsf{Relative}$, Eat, which are binary.

The interpretation of the unary predicates follows their names, $\mathsf{Relative}(x, y)$ represents that "x is a relative of y", while $\mathsf{Eat}(x, y)$ means "x eats y". (5 pts)

b) Show that your formalisation is satisfiable. (3 pts)

4. Consider the following first-order formulas with predicate constants P and R:

$$E \quad (\forall x)(\exists y)(\forall z)(\exists w)\ (\mathsf{R}(x, y) \vee \neg\mathsf{R}(w, z)))$$
$$F \quad ((\forall x)\mathsf{P}(x) \rightarrow (\exists z)\mathsf{R}(z, z)) \rightarrow (\neg(\exists z)\mathsf{R}(z, z) \rightarrow \neg(\forall x)\mathsf{P}(x))$$

a) Give the SNF of the formulas E and F. (4 pts)

b) Decide whether E is valid or not. If not provide an interpretation which falsifies E. Otherwise proof validity by using natural deduction or resolution. (4 pts)

c) Decide whether F is valid or not. If not provide an interpretation which falsifies F. Otherwise proof validity by using natural deduction or resolution. (5 pts)

5. Determine whether the following statements are true or false. Every correct answer is worth 1 points (and every wrong -1 points). (10 pts)

- An interpretation $\mathcal{I}$ is a pair $\mathcal{A} = (A, a)$ such that (i) A is a non-empty set, called *domain* and (ii) the mapping a associates constants with the domain.
- For all formulas F and all sets of formulas $\mathcal{G}$ we have that $\mathcal{G} \models F$ iff $\neg\,\mathsf{Sat}(\mathcal{G} \cup \{\neg F\})$.
- Let $\mathcal{A}$, $\mathcal{B}$ be structures such that $\mathcal{A} \cong \mathcal{B}$. Then for every sentence F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.
- If formula G is obtained from formula F on replacing a subsentence A by an equivalent sentence B then F and G are equivalent.
- If a set of formulas $\mathcal{G}$ has arbitrarily large finite models, then it has a countable infinite model.
- Reachability is not expressible by a single first-order sentence, but with an infinite set of sentences.
- Reachability is expressible in second-order logic.
- Second-order logic is axiomatisable, that is, there exists a proof system for second-order logic that is sound and complete.
- Let $\mathcal{K}$ be a set of finite structures. A sentence F is equivalent to a sentence in existential second-order logic iff F-$\mathcal{K} \in \mathsf{NP}$.
- There exists no satisfiable set of first-order sentences $\mathcal{G}$, such that there exists no Herbrand model of $\mathcal{G}$.