Computational Logic

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Summary Last Lecture

Compactness Theorem

if every finite subset of a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a model

Löwenheim-Skolem Theorem

if a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a countable model

compactness

Löwenheim-Skolem

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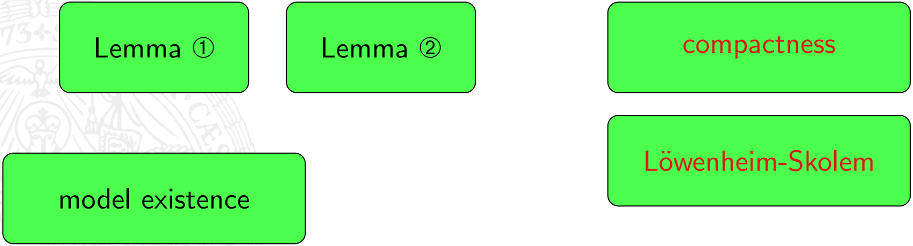

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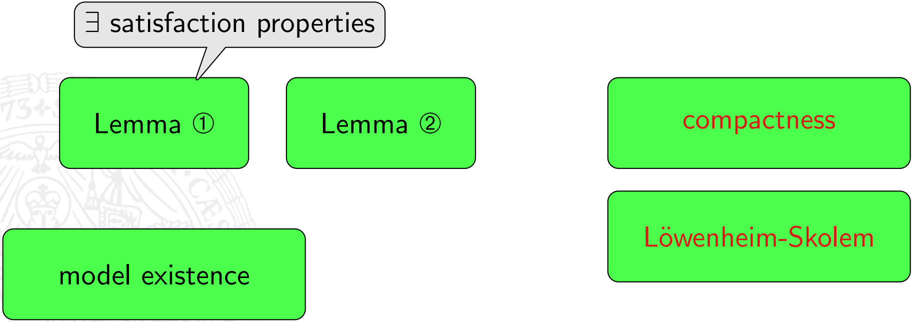

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S admits satisfaction properties $\Rightarrow$ S^* admits satisfaction properties

Lemma ①

Lemma ②

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model existence

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S admits satisfaction properties $\Rightarrow$ $\mathcal{G} \in S$ is satisfiable

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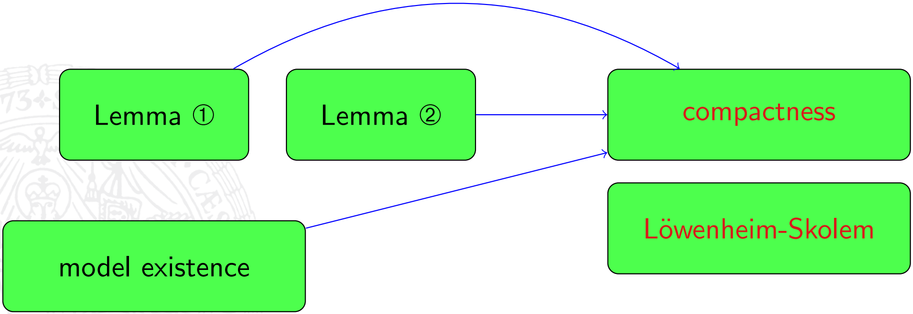

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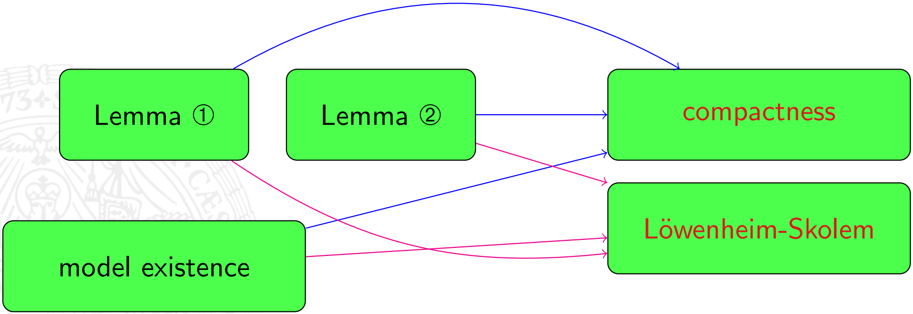

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Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, undecidability of first-order, Löwenheim-Skolem, compactness, model existence theorem, natural deduction, completeness

Extensions and Restrictions of First Order

intuitionistic logic, modal logic, second-order logic

Automated Reasoning

normal forms, Herbrand's Theorem, history of theorem proving, automated reasoning (with equality)

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Theorem (Model Existence Theorem)

1. *if S^* is a set of formula sets of $\mathcal{L}^+$ having the satisfaction properties, then $\forall$ formula sets $\mathcal{G} \in S^*$ of $\mathcal{L}$, $\exists$ $\mathcal{M}$, $\mathcal{M} \models \mathcal{G}$*
2. *$\forall$ elements m of $\mathcal{M}$: m denotes term in $\mathcal{L}^+$*

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2. *$\forall$ elements m of $\mathcal{M}$: m denotes term in $\mathcal{L}^+$*

$\mathcal{G}$ has closure properties $\Rightarrow$ $\exists$ model $\mathcal{M}$, $\mathcal{M} \models \mathcal{G}$

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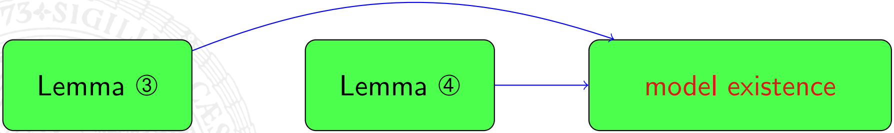

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2. *if $\neg\neg F \in \mathcal{G}$, then $F \in \mathcal{G}$*
3. *if $(E \vee F) \in \mathcal{G}$, then $E \in \mathcal{G}$ or $F \in \mathcal{G}$*

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4. *if $\neg(E \vee F) \in \mathcal{G}$, then $\neg E \in \mathcal{G}$ and $\neg F \in \mathcal{G}$*
5. *if $\exists x F(x) \in \mathcal{G}$, then $\exists$ term t (of $\mathcal{L}^+$), $F(t) \in \mathcal{G}$*

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the set $\mathcal{G}$ of formulas that are true in $\mathcal{M}$ admit

1. no formula F and $\neg F$ in $\mathcal{G}$
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3. if $(E \vee F) \in \mathcal{G}$, then $E \in \mathcal{G}$ or $F \in \mathcal{G}$
4. if $\neg(E \vee F) \in \mathcal{G}$, then $\neg E \in \mathcal{G}$ and $\neg F \in \mathcal{G}$
5. if $\exists x F(x) \in \mathcal{G}$, then $\exists$ term t (of $\mathcal{L}^+$), $F(t) \in \mathcal{G}$
6. if $\neg\exists x F(x) \in \mathcal{G}$, then $\forall$ term t (of $\mathcal{L}^+$), $\neg F(t) \in \mathcal{G}$

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Definition

we call the properties of $\mathcal{G}$ closure properties

Lemma ③

1. let $\mathcal{G}$ be a formula set admitting the closure properties

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Proof of Model Existence

by Lemma ④ and Lemma ③ ∎

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(no identity, no function symbols)

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Proof

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$$c^{\mathcal{M}} := c$$

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3. $\forall$ predicate constant P, $\forall$ terms $t_1, \dots, t_n$:

$$(t_1, \dots, t_n) \in P^{\mathcal{M}} \iff P(t_1, \dots, t_n) \in \mathcal{G}$$

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  by assumption $\mathcal{G}$ fulfils the closure properties, hence there exists a term t such that $G(t) \in \mathcal{G}$

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  by induction hypothesis: $\mathcal{M} \models G(t)$

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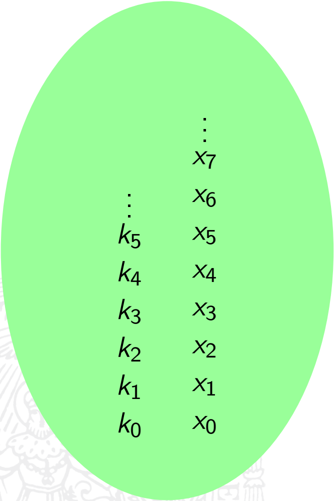
Model Construction in a Picture

set of terms over $\mathcal{L}$



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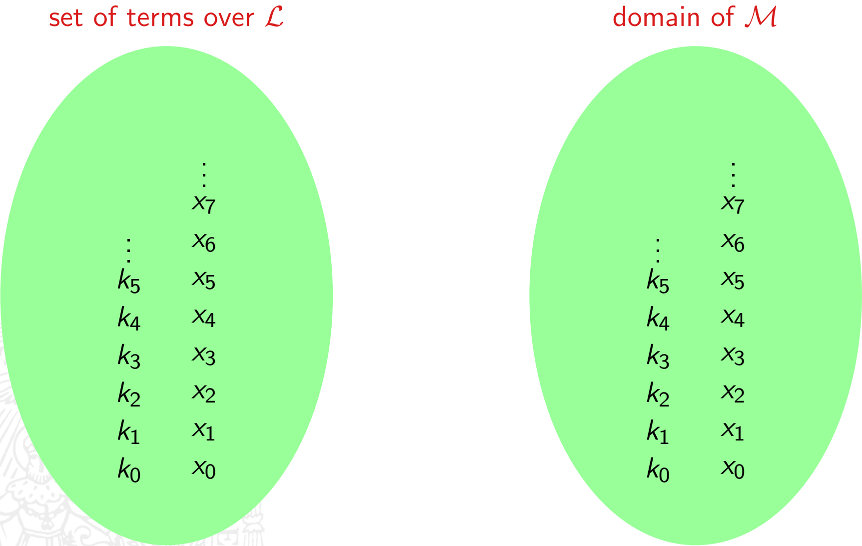

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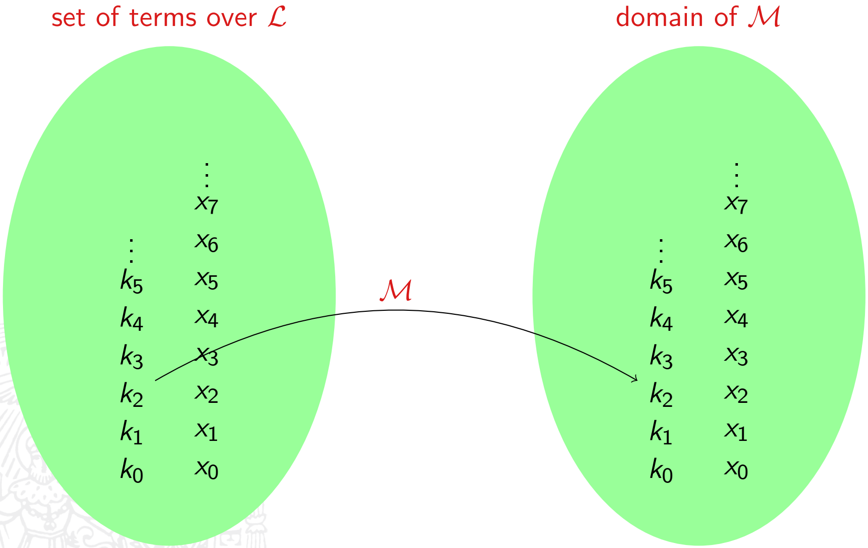

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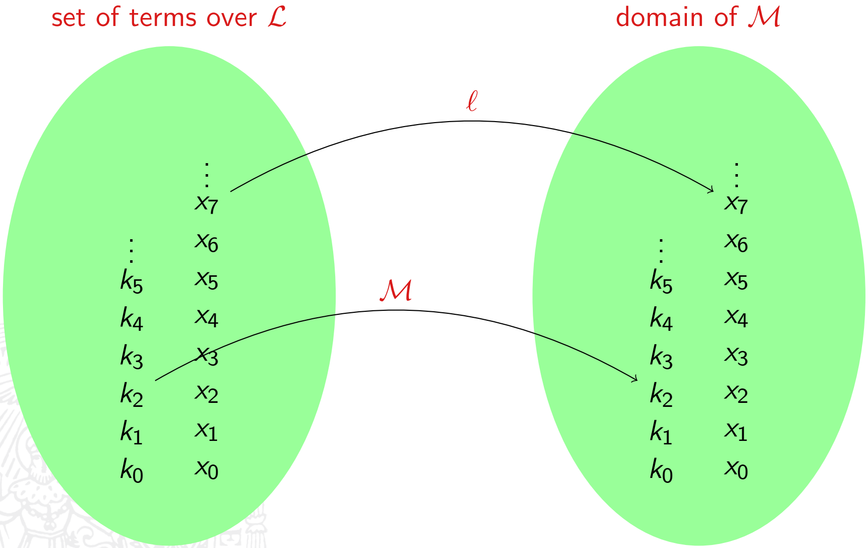

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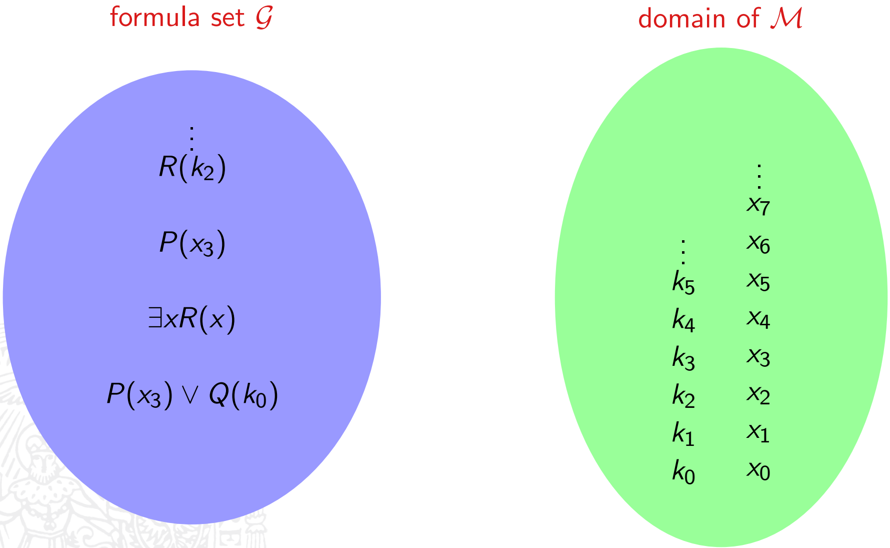


Model Construction in a Picture

formula set $\mathcal{G}$

$\vdots$

$R(k_2)$

$P(x_3)$

$\exists x R(x)$

$P(x_3) \vee Q(k_0)$

domain of $\mathcal{M}$

$k_2 \in R^{\mathcal{M}}$

$\vdots$

$x_3 \in P^{\mathcal{M}}$ x_7

$\vdots$ x_6

k_5 x_5

k_4 x_4

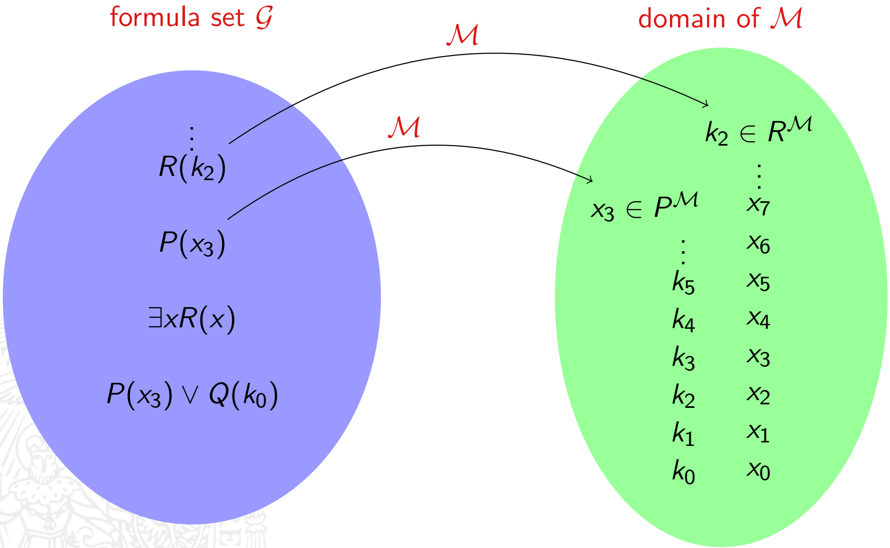
k_3 x_3

k_2 x_2

k_1 x_1

k_0 x_0

Model Construction in a Picture



Proof of Lemma ④

(no identity, no function symbols)

- let $\mathcal{L}$ be a language; $\mathcal{L}^+$ extension of $\mathcal{L}$ with infinitely many individual constants
- let S^* be a set of formula sets (of $\mathcal{L}^+$), let S^* admit the satisfaction properties
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Proof

- construct sequence of sets belonging to S^*

$$\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots \qquad \mathcal{G}_n \subseteq \mathcal{G}_{n+1}$$

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$$\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots \qquad \mathcal{G}_n \subseteq \mathcal{G}_{n+1}$$

- $\mathcal{G}_n$ is constructed in step n

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- $\forall$ formula set $\mathcal{G} \in S^*$ (of $\mathcal{L}$), $\exists\ \mathcal{G}^* \supseteq \mathcal{G}$ (of $\mathcal{L}^+$), such that $\mathcal{G}^*$ fulfils the closure properties

Proof

- construct sequence of sets belonging to S^*

$$\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots \qquad \mathcal{G}_n \subseteq \mathcal{G}_{n+1}$$

- $\mathcal{G}_n$ is constructed in step n
- set $\mathcal{G}^* = \bigcup_{n \geqslant 0} \mathcal{G}_n$

Proof of Lemma ④

(no identity, no function symbols)

- let $\mathcal{L}$ be a language; $\mathcal{L}^+$ extension of $\mathcal{L}$ with infinitely many individual constants
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- $\mathcal{G}_n$ is constructed in step n
- set $\mathcal{G}^* = \bigcup_{n \geqslant 0} \mathcal{G}_n$
- closure properties induce (infinitely many) demands

Proof (cont'd)

Demands

1. no formula F and $\neg F$ in $\mathcal{G}_n$ for all $n \geqslant 0$

Proof (cont'd)

Demands

1. no formula F and $\neg F$ in $\mathcal{G}_n$ for all $n \geqslant 0$
2. if $\neg\neg F \in \mathcal{G}_n$, then $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F\}$

Proof (cont'd)

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2. if $\neg\neg F \in \mathcal{G}_n$, then $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F\}$
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Proof (cont'd)

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5. if $\exists x F(x) \in \mathcal{G}_n$, then $\exists$ term t, $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F(t)\}$

Proof (cont'd)

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Claim: all demands can be granted

Proof (cont'd)

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Claim: all demands can be granted

- invariant of construction: $\forall n \geqslant 0$ we have $\mathcal{G}_n \in S^*$

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- invariant of construction: $\forall n \geqslant 0$ we have $\mathcal{G}_n \in S^*$
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Claim: all demands can be granted

- invariant of construction: $\forall n \geqslant 0$ we have $\mathcal{G}_n \in S^*$
- invariant takes care of first demand: no formula F and $\neg F$ in $\mathcal{G}_n$ for all $n \geqslant 0$
- the satisfaction properties guarantee that any demand can be met

Proof (cont'd)

Proof (cont'd)

- consider Demand 5:
  if $\exists x F(x) \in \mathcal{G}_n$, then $\exists$ term t, $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F(t)\}$

Proof (cont'd)

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- we use that S^* fulfils the satisfaction properties (c is fresh):

$$\exists x F(x) \in \mathcal{G}_n \in S^* \Rightarrow \mathcal{G}_n \cup \{F(c)\} \in S^*$$

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- we fulfil demand by setting (at step k)

$$\mathcal{G}_{k+1} := \mathcal{G}_k \cup \{F(c)\} \qquad \text{for fresh } c$$

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- similar for the Demands 2–6

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Claim: $\exists$ fair strategy

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- assign a pair (i, n) to each demand:
  i is the number of the demand raised at step n

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Claim: $\exists$ fair strategy

- assign a pair (i, n) to each demand:
  i is the number of the demand raised at step n
- enumerate all pairs (i, n), that is, encode (i, n) as number k

Proof (cont'd)

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- similar for the Demands 2–6

Claim: $\exists$ fair strategy

- assign a pair (i, n) to each demand:
  i is the number of the demand raised at step n
- enumerate all pairs (i, n), that is, encode (i, n) as number k
- in step k we grant the i^{th} demand that was raised at step n

Proof (cont'd)

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Saturation of $\mathcal{G}$ in a Picture

formula set $\mathcal{G} = \mathcal{G}_0$

$$\vdots$$

$$\neg\neg T(k_0, k_1)$$

$$\exists x R(x)$$

$$P(x_3) \vee Q(k_0)$$

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$T(k_0, k_1)$

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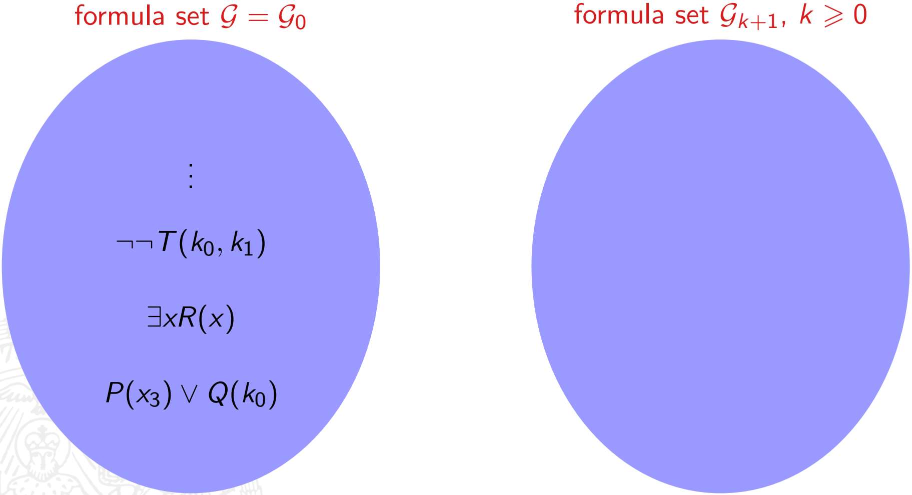

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$\vdots$

$R(k_2)$

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Natural Deduction for First-Order Logic

	introduction	elimination
$\land$	$\dfrac{E \quad F}{E \land F}\ \land\!: \mathrm{i}$	$\dfrac{E \land F}{E}\ \land\!: \mathrm{e} \qquad \dfrac{E \land F}{F}\ \land\!: \mathrm{e}$
$\lor$	$\dfrac{E}{E \lor F}\ \lor\!: \mathrm{i} \qquad \dfrac{F}{E \lor F}\ \lor\!: \mathrm{i}$	$\dfrac{E \lor F \quad \boxed{\begin{array}{c} E \\ \vdots \\ G \end{array}} \quad \boxed{\begin{array}{c} F \\ \vdots \\ G \end{array}}}{G}\ \lor\!: \mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c} E \\ \vdots \\ F \end{array}}}{E \rightarrow F}\ \rightarrow\!: \mathrm{i}$	$\dfrac{E \quad E \rightarrow F}{F}\ \rightarrow\!: \mathrm{e}$

Natural Deduction Extended

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{matrix} E \\ \vdots \\ \bot \end{matrix}}}{\neg E}$ $\neg$: i	$\dfrac{F \quad \neg F}{\bot}$ $\neg$: e
$\neg$		$\dfrac{\bot}{F}$ $\neg$: e
$\neg\neg$		$\dfrac{\neg\neg F}{F}$ $\neg\neg$: e
$=$	$\dfrac{}{t = t}$ $=$: i	$\dfrac{s = t \quad F(s)}{F(t)}$ $=$: e

Natural Deduction Quantifier Rules

	introduction	elimination
$\exists$	$\dfrac{F(t)}{\exists x F(x)}\ \exists\!: \mathrm{i}$	$\dfrac{\exists x F(x) \quad \boxed{\begin{array}{ll} x & F(x) \\ & \vdots \\ & G \end{array}}}{G}\ \exists\!: \mathrm{e}$
$\forall$	$\dfrac{\boxed{\begin{array}{ll} x & \\ & \vdots \\ & F(x) \end{array}}}{\forall x F(x)}\ \forall\!: \mathrm{i}$	$\dfrac{\forall x F(x)}{F(t)}\ \forall\!: \mathrm{e}$

variable x in $\exists$: e, $\forall$: i local to box

Example

1		$\exists x P(x)$	premise
2		$\forall x \forall y (P(x) \rightarrow Q(y))$	premise
3	y		
4	x	$P(x)$	assumption
5		$\forall y (P(x) \rightarrow Q(y))$	$2, \forall$: e
6		$P(x) \rightarrow Q(y)$	$5, \forall$: e
7		$Q(y)$	$4, 6, \rightarrow$: e
8		$Q(y)$	$1, 4-7, \exists$: e
9		$\forall y Q(y)$	$3-8, \forall$: i

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hence we have

$$\exists x P(x), \forall x \forall y (P(x) \rightarrow Q(y)) \vdash \forall y Q(y)$$

Example

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2		$\forall x \forall y (P(x) \to Q(y))$	premise
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4	x	$P(x)$	assumption
5		$\forall y (P(x) \to Q(y))$	$2, \forall$: e
6		$P(x) \to Q(y)$	$5, \forall$: e
7		$Q(y)$	$4, 6, \to$: e
8		$Q(y)$	$1, 4-7, \exists$: e
9		$\forall y Q(y)$	$3-8, \forall$: i

hence we have

provability relation

$$\exists x P(x), \forall x \forall y (P(x) \to Q(y)) \vdash \forall y Q(y)$$

Gödel's Completeness Theorem

Definition

let $\mathcal{G}$ be a set of formulas, F a formula

- if $\exists$ a natural deduction proof from of F from finite $\mathcal{G}_0 \subseteq \mathcal{G}$, we write $\mathcal{G} \vdash F$

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Theorem

first-order logic is sound and complete: $\mathcal{G} \models F \iff \mathcal{G} \vdash F$

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let $\mathcal{G}$ be a set of formulas, F a formula

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- if $\neg \exists$ proof of $\bot$ from $\mathcal{G}$, we say $\mathcal{G}$ is consistent, otherwise inconsistent

Theorem

first-order logic is sound and complete: $\mathcal{G} \models F \iff \mathcal{G} \vdash F$

Proof Idea

- the set S of consistent sets of formulas admit the satisfactions properties

Gödel's Completeness Theorem

Definition

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first-order logic is sound and complete: $\mathcal{G} \models F \iff \mathcal{G} \vdash F$

Proof Idea

- the set S of consistent sets of formulas admit the satisfactions properties
- by the model existence theorem any $\mathcal{G} \in S$ is satisfiable ∎

Soundness Theorem

first-order logic is sound

$$\mathcal{G} \models F \Leftarrow \mathcal{G} \vdash F$$

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Lemma ⑥

model existence

completeness

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S set of consistent sets $\Rightarrow$ S admits satisfaction properties

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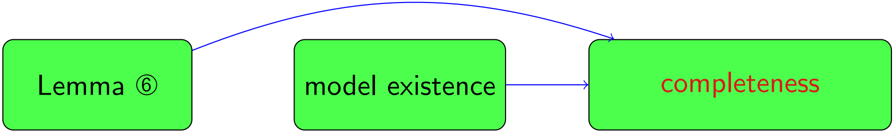

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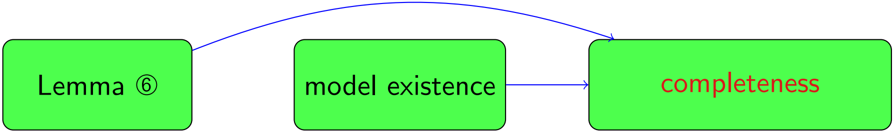

Lemma ⑥

the set S of all consistent set of formulas has the satisfaction properties

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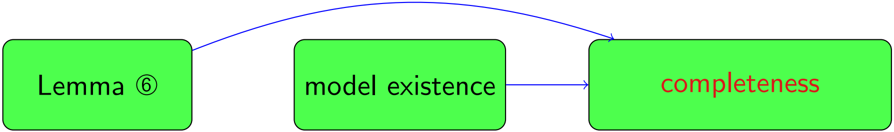

Lemma

$\mathcal{G} \vdash F$ *iff* $\mathcal{G} \cup \{\neg F\}$ *is inconsistent*

Lemma ⑥

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we have only considered the case without function symbols, without $=$