Functional Programming

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week 6

λ-Calculus

λ-Terms

$$t ::= \overbrace{x}^{\text{Variable}} \mid \underbrace{(\lambda x.t)}_{\text{Abstraction}} \mid \overbrace{(t\ t)}^{\text{Application}}$$

Example

$x\ y$	$(x\ y)$	"x applied to y"
$\lambda x.x$	$(\lambda x.x)$	"lambda x to x"
$\lambda xy.x$	$(\lambda x.(\lambda y.x))$	"lambda x y to x"
$\lambda x.x\ x$	$(\lambda x.(x\ x))$	"lambda x to (x applied to x)"
$(\lambda x.x)\ x$	$((\lambda x.x)\ x)$	"(lambda x to x) applied to x"

λ-Calculus (cont'd)

β-Reduction

the term s (β-)reduces to the term t in one step, i.e.,

$$\overbrace{s \to_\beta t}^{(\beta\text{-})\text{step}}$$

iff there exist context C and terms u, v s.t.

$$s = C[(\lambda x.u)\ v] \qquad \text{and} \qquad t = C[u\{x/v\}]$$

Example

$$K \stackrel{\text{def}}{=} \lambda xy.x$$

$$I \stackrel{\text{def}}{=} \lambda x.x$$

$$\Omega \stackrel{\text{def}}{=} (\lambda x.x\ x)\ (\lambda x.x\ x)$$

This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction, reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, monads, ...

Booleans and Conditionals

OCaml

- `true`
- `false`
- `if` b `then` t `else` e

λ-Calculus

- $\mathsf{true} \stackrel{\text{def}}{=} \lambda xy.x$
- $\mathsf{false} \stackrel{\text{def}}{=} \lambda xy.y$
- $\mathsf{if} \stackrel{\text{def}}{=} \lambda xyz.x\ y\ z$

Example

$$\mathsf{if}\ \mathsf{true}\ t\ e \rightarrow_\beta^+ \mathsf{true}\ t\ e \rightarrow_\beta^+ t$$
$$\mathsf{if}\ \mathsf{false}\ t\ e \rightarrow_\beta^+ \mathsf{false}\ t\ e \rightarrow_\beta^+ e$$

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	$\overline{\mathsf{n}} \stackrel{\text{def}}{=} \lambda fx.f^n\ x$
`( + )`	$\mathsf{add} \stackrel{\text{def}}{=} \lambda mnfx.m\ f\ (n\ f\ x)$
`( * )`	$\mathsf{mul} \stackrel{\text{def}}{=} \lambda mnf.m\ (n\ f)$
`( ** )`	$\mathsf{exp} \stackrel{\text{def}}{=} \lambda mn.n\ m$

Example

$$\mathsf{add}\ \overline{1}\ \overline{1} \rightarrow_\beta^* \overline{2}$$

Pairs

OCaml vs. λ-Calculus

`fun x y -> (x,y)`	$\text{pair} \stackrel{\text{def}}{=} \lambda xyf.f\ x\ y$
`fst`	$\text{fst} \stackrel{\text{def}}{=} \lambda p.p\ \text{true}$
`snd`	$\text{snd} \stackrel{\text{def}}{=} \lambda p.p\ \text{false}$

Example

$$\text{fst}\ (\text{pair}\ \overline{\text{m}}\ \overline{\text{n}}) \to_\beta^* \overline{\text{m}}$$

Lists

OCaml vs. λ-Calculus

`::`	$\text{cons} \stackrel{\text{def}}{=} \lambda xy.\text{pair false}\ (\text{pair}\ x\ y)$
`hd`	$\text{hd} \stackrel{\text{def}}{=} \lambda z.\text{fst}\ (\text{snd}\ z)$
`tl`	$\text{tl} \stackrel{\text{def}}{=} \lambda z.\text{snd}\ (\text{snd}\ z)$
`[]`	$\text{nil} \stackrel{\text{def}}{=} \lambda x.x$
`fun x -> x = []`	$\text{null} \stackrel{\text{def}}{=} \text{fst}$

Example

$$\text{null nil} \to_\beta^* \text{true}$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                              else 1 + length(tl x)
```

λ-Calculus

$$\text{length} \stackrel{\text{def}}{=} \mathsf{Y}\ (\lambda f x.\text{if } (\text{null } x)\ \overline{0}\ (\text{add } \overline{1}\ (f\ (\text{tl } x))))$$

Definition (Y-combinator)

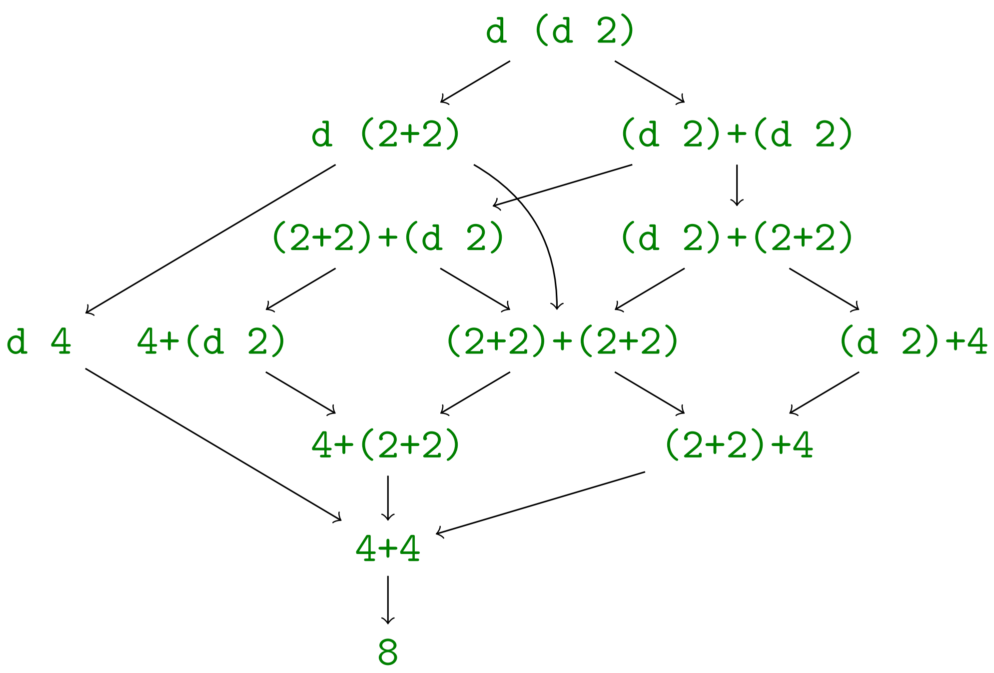
$$\mathsf{Y} \stackrel{\text{def}}{=} \lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))$$

Y has fixed point property, i.e., for all $t \in \mathcal{T}(\mathcal{V})$

$$\mathsf{Y}\ t \leftrightarrow^* t\ (\mathsf{Y}\ t)$$

Example

- consider `let d x = x + x`
- the term `d (d 2)` can be evaluated as follows (9 possibilities)



Strategies

Strategy

- fixes evaluation order
- examples: call-by-value and call-by-name

Example

```
let d x = x + x
```

- call-by-value:

$$\begin{aligned} \texttt{d (d 2)} &\to \texttt{d (2+2)} \\ &\to \texttt{d 4} \\ &\to \texttt{4 + 4} \\ &\to \texttt{8} \end{aligned}$$

- call-by-name:

$$\begin{aligned} \texttt{d (d 2)} &\to \texttt{(d 2)+(d 2)} \\ &\to \texttt{(2+2)+(d 2)} \\ &\to \texttt{4+(d 2)} \\ &\to \texttt{4+(2+2)} \\ &\to \texttt{4+4} \\ &\to \texttt{8} \end{aligned}$$

(Leftmost) Innermost Reduction

- always reduce leftmost innermost redex

Definition

redex t of term u is innermost if it does not contain a redex as proper subterm, i.e.,

$$\nexists s \in \mathcal{S}\text{ub}(t) \text{ s.t. } s \neq t \text{ and } s \text{ is a redex}$$

Example

Consider $t = (\lambda x.(\lambda y.y)\ x)\ z$

- $(\lambda y.y)\ x$ is innermost redex
- $(\lambda x.(\lambda y.y)\ x)\ z$ is redex, but not innermost

(Leftmost) Outermost Reduction

- always reduce leftmost outermost redex

Definition

redex t of term u is outermost if it is not a proper subterm of some other redex in u, i.e.,

$$\nexists s \in \mathcal{S}\text{ub}(u) \text{ s.t. } s \text{ is a redex and } t \in \mathcal{S}\text{ub}(s) \text{ and } s \neq t$$

Example

Consider $t = (\lambda x.(\lambda y.y)\ x)\ z$

- $(\lambda x.(\lambda y.y)\ x)\ z$ is outermost redex
- $(\lambda y.y)\ x$ is redex, but not outermost

Call-by-Value

- use innermost reduction
- corresponds to strict (or eager) evaluation, e.g., OCaml
- slight modification: only reduce terms that are not in WHNF

Definition (Weak head normal form)

term t is in weak head normal form ($WHNF$) iff

$$t \neq u\ v$$

Example (WHNF)

$\lambda x.x$ ✓ $(\lambda x.x)\ y$ × $(\lambda x.x)\ y\ z$ × $\lambda x.(\lambda y.y)\ x$ ✓ $x\ x$ ×

Call-by-Name

- use outermost reduction
- corresponds to lazy evaluation (without memoization), e.g., Haskell
- slight modification: only reduce terms that are not in WHNF

λ Tree Tool

developed by Stefan Widerin in bachelor project

λ-Terms

$$t ::= \texttt{x} \mid (\texttt{\textbackslash x.}t) \mid (t\ t)$$

Conventions

- nested abstractions use spaces to separate variable names, e.g.,

$$\lambda xy.x \quad \texttt{\textbackslash x y.x}$$
$$\lambda x_1.y \quad \texttt{\textbackslash x1.y}$$

Result

Animator

- animation of β-steps
- innermost/outermost
- leftmost/rightmost

Output

- reduction sequence to NF
- innermost/outermost
- leftmost/rightmost