

# Functional Programming

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week 7

Week 7 - Induction

Summary of Week 6

## Rewrite Strategies

### Outermost

- ▶ choose the (leftmost) outermost redex
- ▶ redex is **outermost** if not subterm of different redex

### Innermost

- ▶ choose the (leftmost) innermost redex
- ▶ redex is **innermost** if no proper subterm is redex

# Reduction Strategies

## Call-by-name

- ▶ use outermost strategy
- ▶ stop as soon as WHNF is reached

## Call-by-value

- ▶ use innermost strategy
- ▶ stop as soon as WHNF is reached

## WHNF (Intuition)

*Thou shalt not reduce below lambda.*

# Evaluation Strategies

## Lazy

- ▶ call-by-name + sharing
- ▶ only evaluate if necessary
- ▶ e.g. Haskell

## Strict/Eager

- ▶ call-by-value
- ▶ evaluate arguments before calling a function
- ▶ e.g. OCaml (also support for laziness)

## This Week

### Practice I

OCaml introduction, lists, strings, trees

### Theory I

lambda-calculus, evaluation strategies, induction,  
reasoning about functional programs

### Practice II

efficiency, tail-recursion, combinator-parsing

### Theory II

type checking, type inference

### Advanced Topics

lazy evaluation, infinite data structures, monads, ...

## When?

### Goal

*“prove that some property  $P$  holds for all natural numbers”*

### Formally

$$\forall n. P(n) \quad (\text{where } n \in \mathbb{N})$$

## How?

2 goals to show

1.  $P(0)$
2.  $\forall k.(P(k) \rightarrow P(k + 1))$

Gives

$$(P(0) \wedge \forall k.(P(k) \rightarrow P(k + 1))) \rightarrow \forall n.P(n)$$

## Why Does This Work?

We have

- ▶  $P(0)$  “property  $P$  holds for 0”
- ▶  $\forall k.(P(k) \rightarrow P(k + 1))$  “if property  $P$  holds for arbitrary  $k$  then it also holds for  $k + 1$ ”

We want

$\forall n.P(n)$  “ $P$  holds for every  $n$ ”

We get

- |                                |                                    |
|--------------------------------|------------------------------------|
| ▶ for the moment fix $n$       | ▶ ...                              |
| ▶ have $P(0)$                  | ▶ have $P(n - 1)$                  |
| ▶ have $P(0) \rightarrow P(1)$ | ▶ have $P(n - 1) \rightarrow P(n)$ |
| ▶ have $P(1)$                  | ▶ hence $P(n)$                     |
| ▶ have $P(1) \rightarrow P(2)$ |                                    |

## What is Meant by 'Property'?

anything that depends on some variable and is either true or false can be seen as function  $p : 'a \rightarrow \text{bool}$

### Example

- ▶  $P(x) = (1 + 2 + \dots + x = \frac{x \cdot (x+1)}{2})$
- ▶ base case:  $P(0) = (1 + 2 + \dots + 0 = 0 = \frac{0 \cdot (0+1)}{2})$
- ▶ step case:  $P(k) \rightarrow P(k + 1)$   
 IH:  $P(k) = (1 + 2 + \dots + k = \frac{k \cdot (k+1)}{2})$   
 show:  $P(k + 1)$

$$\begin{aligned}
 1 + 2 + \dots + (k + 1) &= (1 + 2 + \dots + k) + (k + 1) \\
 &\stackrel{\text{IH}}{=} \frac{k \cdot (k + 1)}{2} + (k + 1) \\
 &= \frac{(k + 1) \cdot (k + 2)}{2}
 \end{aligned}$$

## Remark

- ▶ of course the base case can be changed
- ▶ e.g., if base case  $P(1)$ , property holds for all  $n \geq 1$

# Recall

## Type

```
type 'a list = [] | (:::) of 'a * 'a list
```

## Note

- ▶ lists are recursive structures
- ▶ base case: []
- ▶ step case:  $x :: xs$

# Induction Principle on Lists

## Intuition

- ▶ to show  $P(xs)$  for all lists  $xs$
- ▶ show base case:  $P([])$
- ▶ show step case:  $P(xs) \rightarrow P(x :: xs)$  for arbitrary  $x$  and  $xs$

## Formally

$$(P([]) \wedge \forall x : \alpha. \forall xs : \alpha \text{ list. } \overbrace{P(xs) \rightarrow P(x :: xs)}^{\text{IH}})) \rightarrow \forall ls : \alpha \text{ list. } P(ls)$$

## Remarks

- ▶  $y : \beta$  reads '*y is of type  $\beta$* '
- ▶ for lists,  $P$  can be seen as function  $p : 'a \text{ list} \rightarrow \text{bool}$

## Example - Lst.append

Recall

```
let rec (@) xs ys = match xs with
| []      -> ys
| x::xs  -> x :: (xs @ ys)
```

Lemma

`[]` is *right identity* of `@`, i.e.,

$$xs @ [] = xs$$

Proof.

Blackboard



## Example - Lst.length

Recall

```
let rec length = function []      -> 0
| _::xs  -> 1 + length xs
```

Lemma

*sum of lengths equals length of combined list, i.e.,*

$$\text{length } xs + \text{length } ys = \text{length}(xs @ ys)$$

Proof.

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# General Structures

## Type

```
type term = Var of var
          | Abs of (var * term)
          | App of (term * term)
```

## Induction Principle

- ▶ for every non-recursive constructor there is a base case
  - ▶ base case: `Var x`
- ▶ for every recursive constructor there is a step case
  - ▶ step case: `Abs(x, t)`
  - ▶ step case: `App(s, t)`

# Induction Principle on General Structures

## Intuition

- ▶ to show  $P(s)$  for all structures  $s$
- ▶ show base cases
- ▶ show step cases



## Recall

### Type

```
type 'a btree = Empty | Node of ('a btree * 'a * 'a btree)
```

### Induction Principle

$$\begin{aligned}
 & (P(\text{Empty}) \wedge \\
 & \forall v : \alpha. \forall l : \alpha \text{ btree}. \forall r : \alpha \text{ btree}. \\
 & ((P(l) \wedge P(r)) \rightarrow P(\text{Node}(l, v, r)))) \\
 & \rightarrow \\
 & \forall t : \alpha \text{ btree}. P(t)
 \end{aligned}$$

## Example - Trees

### Definition (Perfect Binary Trees)

binary tree is **perfect** if all leaf nodes have same depth

```
let rec perfect = function
  | Empty          -> true
  | Node(l,_,r) -> height l = height r && perfect l
                  && perfect r
```

## Example - Trees (cont'd)

Recall

```
let rec height = function
  | Empty          -> 0
  | Node(l,_,r)    -> max (height l) (height r) + 1

let rec size = function Empty          -> 0
                    | Node(l,_,r)    -> size l + size r + 1
```

Lemma

*perfect binary tree  $t$  of height  $n$  has exactly  $2^n - 1$  nodes*

Proof.

To show:  $P(t) = (\text{perfect } t \rightarrow (\text{size } t = 2^{(\text{height } t)} - 1))$

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□