Functional Programming

WS 2012/13

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week 8

Overview

- Week 8 - Efficiency
  - Summary of Week 7
  - Fibonacci Numbers
  - Tupling
  - Tail Recursion

Overview

Mathematical Induction

Induction Principle

$$(P(m) \wedge \forall k \geq m.(P(k) \rightarrow P(k+1))) \rightarrow \forall n \geq m.P(n)$$

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$$(\underbrace{P(m)}_{\text{base case}} \land \forall k \geq m.(P(k) \rightarrow P(k+1))) \rightarrow \forall n \geq m.P(n)$$

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- if a domino falls also its right neighbor falls

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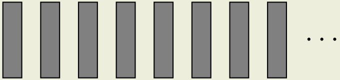

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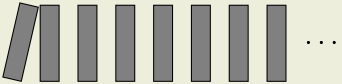

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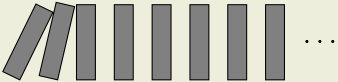

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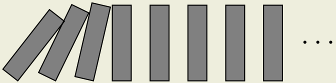

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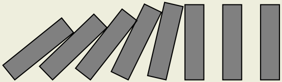

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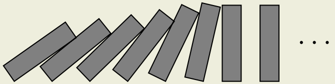

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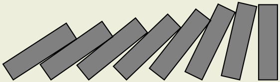

...

Induction on Lists

Induction Principle (without Types)

$$(P([\,]) \wedge \forall x.\forall xs.(P(xs) \to P(x :: xs))) \to \forall ls.P(ls)$$

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Lemma (append is associative)

$$xs \mathbin{@} (ys \mathbin{@} zs) = (xs \mathbin{@} ys) \mathbin{@} zs$$

where

```
let rec (@) xs ys = match xs with
  | []     -> ys
  | x::xs -> x :: (xs @ ys)
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Proof.

Blackboard □

Structural Induction

Usage

- can be used on every variant type
- base cases correspond to non-recursive constructors
- step cases correspond to recursive constructors

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Example

- lists
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- ...

This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction,
reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, monads, ...

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Mathematical

Definition (n-th Fibonacci number)

$$\mathsf{fib}(n) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } n \leq 1 \\ \mathsf{fib}(n-1) + \mathsf{fib}(n-2) & \text{otherwise} \end{cases}$$

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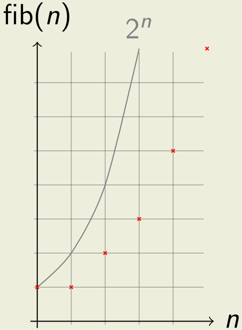
Graph



Mathematical (cont'd)

Example

1

Mathematical (cont'd)

Example

1, 1

Mathematical (cont'd)

Example

1, 1, 2

Mathematical (cont'd)

Example

1, 1, 2, 3

Mathematical (cont'd)

Example

1, 1, 2, 3, 5

Mathematical (cont'd)

Example

1, 1, 2, 3, 5, 8

Mathematical (cont'd)

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1, 1, 2, 3, 5, 8, 13

Mathematical (cont'd)

Example

1, 1, 2, 3, 5, 8, 13, 21

Mathematical (cont'd)

Example

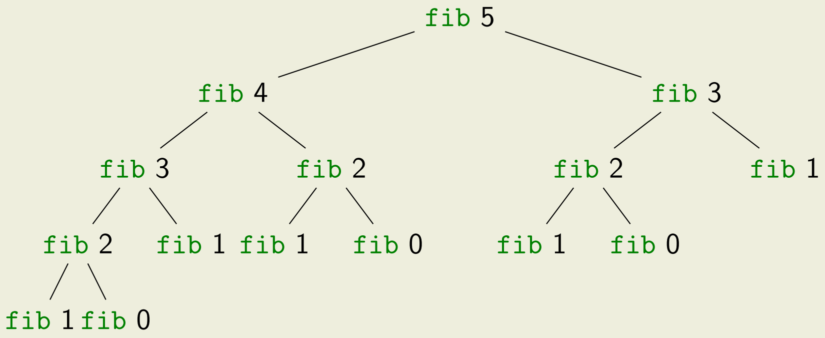
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181 ,6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155, 165580141, 267914296, 433494437, 701408733, 1134903170, 1836311903, 2971215073, . . .

OCaml

Definition

```
let rec fib n = if n < 2 then 1 else fib(n-1) + fib(n-2)
```

Example



Overview

Tupling

Idea

- use tuples to return more than one result
- make results available as return values instead of recomputing them

Fibonacci Numbers

Example

```
let rec fibpair n = if n < 1 then (0,1) else (
 if n = 1 then (1,1)
         else let (f1,f2) = fibpair (n-1) in (f2,f1+f2)
)
```

Fibonacci Numbers

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$$\texttt{fibpair}(n+1) = (\texttt{fib}\ n, \texttt{fib}(n+1))$$

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Lemma

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A Second Example

Goal

compute average value of an integer list

Naive Approach

- `let average xs = IntLst.sum xs / Lst.length xs`

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Combined Function

- ```
  let rec sumlen = function
    | []    -> (0,0)
    | x::xs -> let (sum,len) = sumlen xs in (sum+x,len+1)
  ```
- `let average1 xs = let (sum,len) = sumlen xs in sum/len`
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- one traversal of `xs` suffices

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Recursion vs. Tail Recursion

Idea

- a function calling itself is recursive
- functions that mutually call each other are mutually recursive
- special kind of recursion is tail recursion

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Definition (Tail recursion)

a function is called tail recursive if the last action in the function body is the recursive call

Examples

Length

- ```
  let rec length = function []    -> 0
                          | _::xs -> 1 + length xs
  ```
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Even/Odd

- ```
  let rec is_even = function 0 -> true
                           | 1 -> false
                           | n -> is_odd(n-1)
  and is_odd      = function 0 -> false
                           | 1 -> true
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  ```
- mutually recursive (btw: also tail recursive)

Parameter Accumulation

Idea

- make function tail recursive
- provide data as input instead of computing it before recursive call
- Why? (tail recursive functions can automatically be transformed into space-efficient loops)

Example (Sumlen)

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