Computational Logic

Automated Reasoning

Georg Moser

Institute of Computer Science @ UIBK

Winter 2013

Organisation

Time and Place

Computational Logic	Wednesday, 13:15–15:00	3W03
Automated Theorem Proving	Wednesday, 15:15–17:00	3W03
exercise class	Wednesday, 17:15–18:00	3W03

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week 1	October 2	week 8	November 27
week 2	October 9	week 9	December 4
week 3	October 16	week 10	December 11
week 4	October 23	week 11	January 8
week 5	November 6	week 12	January 15
week 6	November 13	week 13	January 22
week 7	November 20	first exam	January 29

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Office Hours

Thursday, 9:00–11:00, 3M09, IfI Building

Outline of the Module

Advanced Topics in Logic

for example

- compactness
- model existence theorem
- Herbrand's Theorem
- Curry-Howard Isomorphism

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Automated Reasoning

for example

- tableau provers
- redundancy and deletion
- superposition
- Robbins problem

Outline of the Lecture "Computational Logic"

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, Löwenheim-Skolem, compactness, model existence theorem, natural deduction, completeness, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

Outline of the Lecture "Automated Theorem Proving"

Early Approaches in Automated Reasoning

short recollection of Herbrand's theorem, Gilmore's prover, method of Davis and Putnam

Starting Points

resolution, tableau provers, structural Skolemisation, redundancy and deletion

Automated Reasoning with Equality

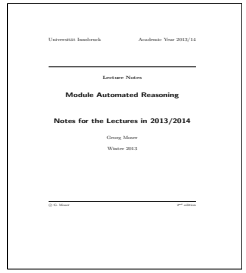
ordered resolution, paramodulation, ordered completion and proof orders, superposition

Applications of Automated Reasoning

Neuman-Stubblebinde Key Exchange Protocol, Robbins problem, resolution and paramodulation as decision procedure, ...

Literature

- lecture notes
  (2nd edition)



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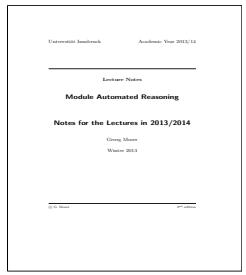

Additional Reading

- G.S. Boolos, J.P. Burgess, and R.C. Jeffrey
  Computability and Logic
  Cambridge University Press, 2007
- H.-D. Ebbinghaus, J. Flum, and W. Thomas
  Einführung in die mathematische Logik
  Spektrum Akademischer Verlag, 2007
- A. Leitsch
  The Resolution Calculus
  Springer-Verlag, 2007
- papers, distributed during the course

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Comments

- officially there are two lectures and one exercise group
- this is nonsense, as the course on theorem proving is based on the course on logic
- suggestion: we start with logic, if we are finished, we continue with theorem proving
- typical scheduling of the block: lecture, exercises, lecture

Introduction

What is Logic?

Argument ①

1. a mother or father of a person is an ancestor of that person
2. an ancestor of an ancestor of a person is an ancestor of a person
3. Sarah is the mother of Isaac, Isaac is the father of Jacob
4. Thus, Sarah is an ancestor of Jacob

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Argument ②

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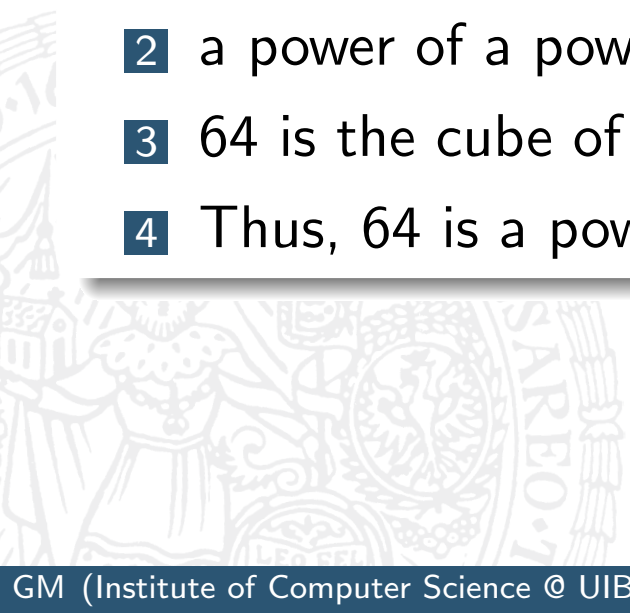

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logic tells us that argument ① = argument ②

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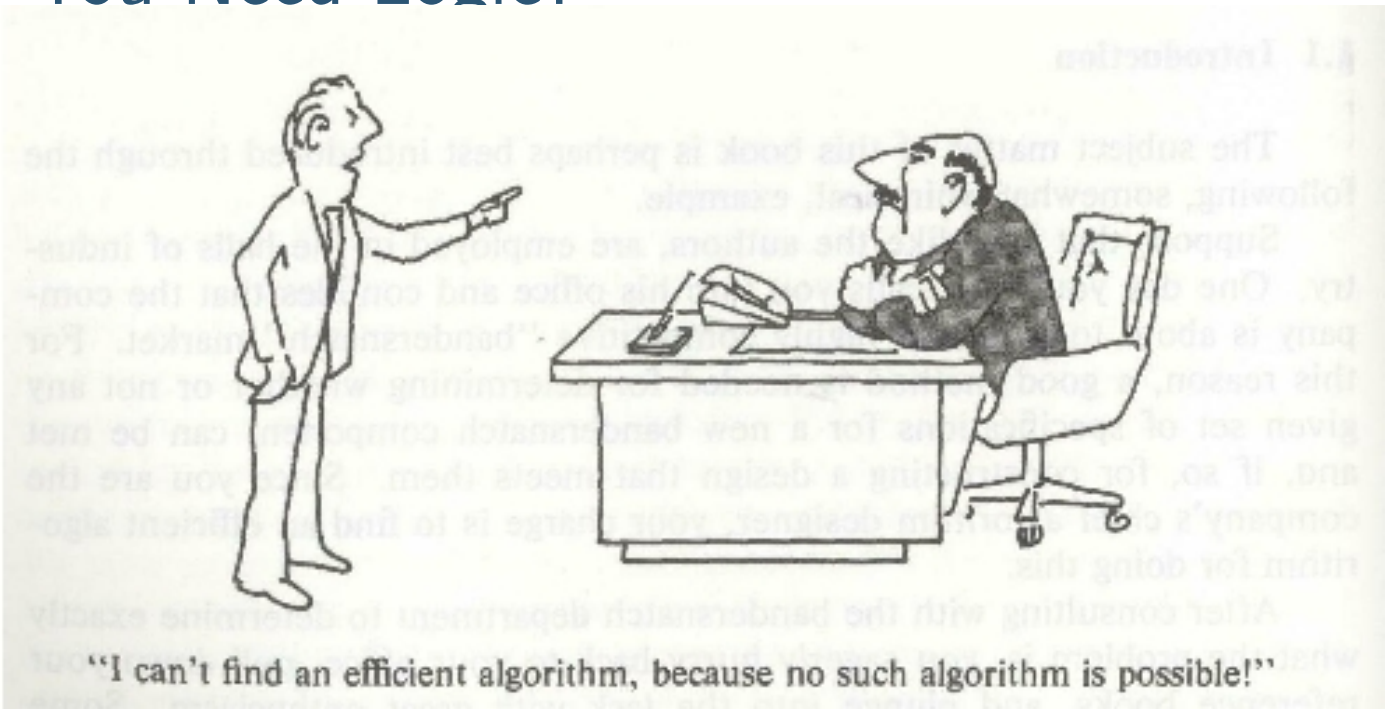

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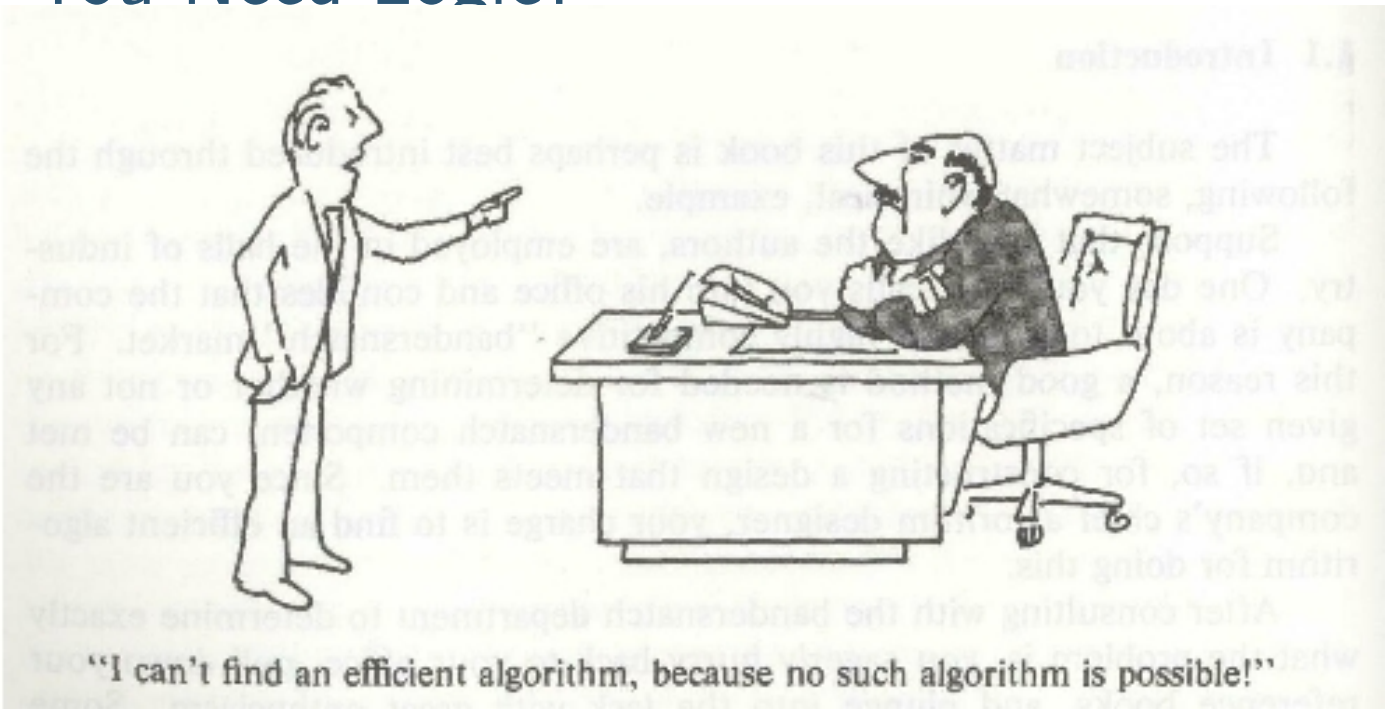

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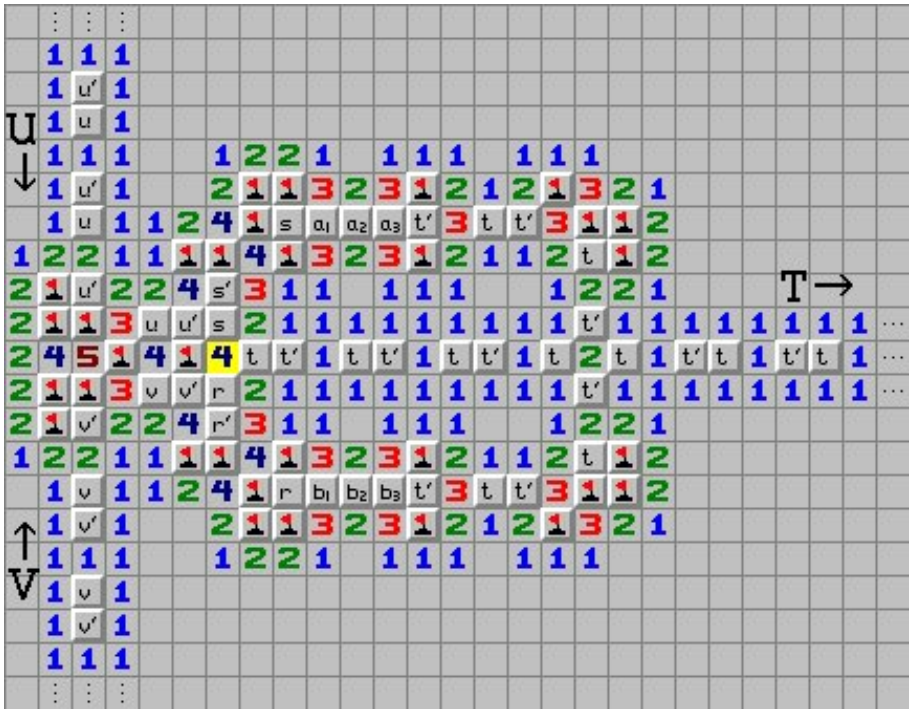
Recall

(for example) the equivalence of programs is undecidable

Proof.

reduction from the undecidability of the *Entscheidungsproblem* (Alonzo Church)

Another Picture



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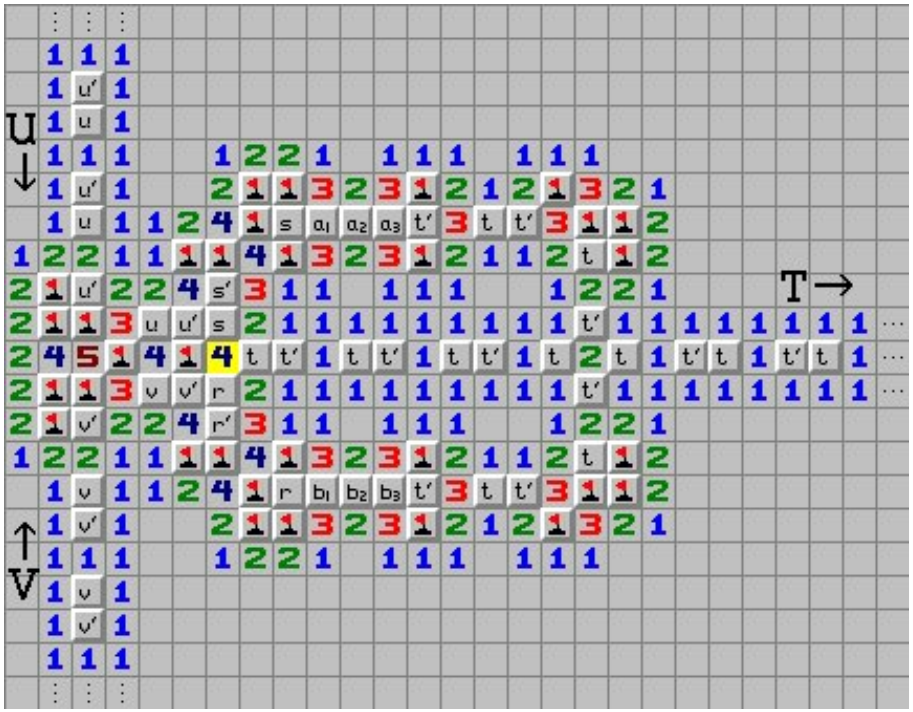

SAT technology

- a Minesweeper solver can be coded by using a SAT solver
- verifying the correctness of a given Minesweeper configuration is an NP-complete problem

A More Serious Answer

Application ①: Program Analysis

- abstract interpretations represent the behaviour of programs
- logical products of interpretations allows the automated combination of simple interpreters
- based on Nelson-Oppen methodology

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Application ②: Databases

- datalog is a declarative language and syntactically it is a subset of Prolog; used in knowledge representation systems
- disjunctive datalog is an extension of datalog that allows disjunctions in heads of rules
- disjunctive datalog is a strict extension of SQL
- (disjunctive) datalog forms the basis of semantic web applications and has connections to description logics and ontologies

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Complexity results

- expression complexity of datalog is EXPTIME-complete
- expression complexity for disjunctive datalog (with $\neg$) is $\text{NEXPTIME}^{\text{NP}}$-complete
- problems complete for $\text{NEXPTIME}^{\text{NP}}$ can (only) be solved on an NTM with NP-oracle running in exponential time

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- alternatively NP can be characterised as the class of existential second-order sentence
- completeness for NP of SAT becomes trivial

A Quiz

Questions

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- can we express the following statement about a given graph $\mathcal{G}$ in first-order logic?

> *let s and t be nodes in $\mathcal{G}$, then there exists a path of length at most 3 from s to t*

BoolTool

- BoolTool is a web-interfaced based tool for manipulation and transformation of formulas in propositional logic
- available at http://cl-informatik.uibk.ac.at/software/booltool/

Propositional Logic

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short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, Löwenheim-Skolem, compactness, model existence theorem, natural deduction, completeness, normalisation

Properties of First Order Logic

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(propositional) formulas are defined as follows

- a propositional atom p and a truth constant is a formula
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we use precedence: $\neg \; > \; \vee, \wedge \; > \; \rightarrow$; right-associativity of $\rightarrow$

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- we write $\mathsf{v}(A)$ for valuation of A, the extension of the assignment to formulas

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Definition

the consequence relation $\models$ asserts that $\mathsf{v}(B) = \mathsf{T}$, whenever $\mathsf{v}(A_1)$, ..., $\mathsf{v}(A_n)$ is true for any assignment v, denoted as $A_1, \ldots, A_n \models B$

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we write $\models A$, instead of $\varnothing \models A$ and call A a tautology or valid

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- we write $\vdash B$ instead of $\varnothing \vdash B$
- B is called provable, if $\vdash B$ holds

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2. study of proof systems led to efficient SAT techniques
3. in designing new logics for applications one starts with the rules, typically the semantics comes later

Soundness & Completeness

Theorem

- $\exists$ *provability relations* $\vdash$ *such that the following holds:*

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Soundness & Completeness

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Example

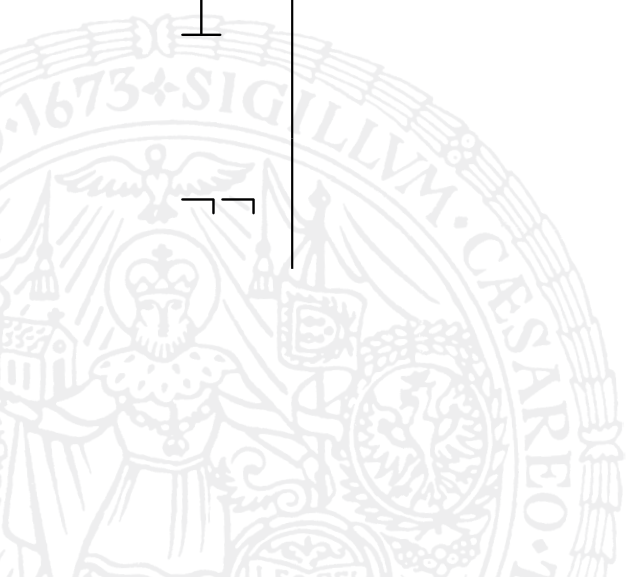
natural deduction for propositional logic

Natural Deduction

	introduction	elimination
$\wedge$	$\dfrac{E \quad F}{E \wedge F}\ \wedge\text{i}$	$\dfrac{E \wedge F}{E}\ \wedge\text{: e} \qquad \dfrac{E \wedge F}{F}\ \wedge\text{: e}$
$\vee$	$\dfrac{E}{E \vee F}\ \vee\text{: i} \qquad \dfrac{F}{E \vee F}\ \vee\text{: i}$	$\dfrac{E \vee F \quad \boxed{\begin{array}{c}E\\ \vdots\\ G\end{array}} \quad \boxed{\begin{array}{c}F\\ \vdots\\ G\end{array}}}{G}\ \vee\text{: e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}E\\ \vdots\\ F\end{array}}}{E \rightarrow F}\ \rightarrow\text{: i}$	$\dfrac{E \quad E \rightarrow F}{F}\ \rightarrow\text{: e}$

Natural Deduction (cont'd)

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{array}{c}E\\ \vdots\\ \bot\end{array}}}{\neg E}\ \neg\text{: i}$	$\dfrac{F \quad \neg F}{\bot}\ \neg\text{: e}$
$\bot$		$\dfrac{\bot}{F}\ \bot\text{: e}$
$\neg\neg$		$\dfrac{\neg\neg F}{F}\ \neg\neg\text{: e}$



Example

derivation of Pierce's law $((p \to q) \to p) \to p$

1	$((p \to q) \to p)$	assumption
2	$\neg p$	assumption
3	p	assumption
4	$\bot$	2, $\neg$ elimination
5	q	$\bot$ elimination
6	$p \to q$	$\to$ introduction
7	p	1, $\to$ elimination
8	$\bot$	2, $\neg$ elimination
9	p	derived rule
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Theorem

let $A \to B$ be valid, $\exists$ C such that $A \to C$, $C \to B$ valid

interpolant C contains only variables that occur in A and B

Propositional Resolution

Definition

- a literal is a propositional atom p or its negation $\neg p$
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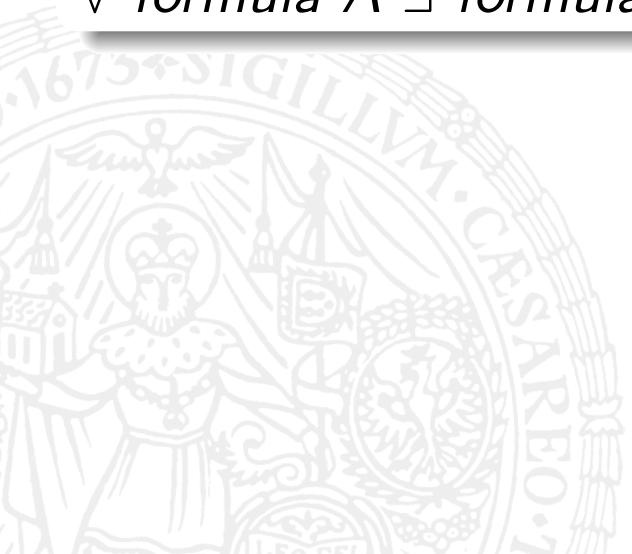

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Definition

a clause is disjunction of literals

- $\square$ is a clause
- literals are clauses
- if C, D are clauses, then $C \vee D$ is a clause

Convention

we use: $p \equiv \neg\neg p$, $\square \vee \square \equiv \square$, $C \vee \square \vee D \equiv C \vee D \equiv D \vee C$

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factoring

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(propositional) resolution is a refutation based technique; not competitive with SAT solvers

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$$\dfrac{\dfrac{\dfrac{p \vee \neg q \vee r \quad \neg p \vee r}{\neg q \vee r \vee r} \qquad \dfrac{q \vee r \qquad \neg q \vee r}{r \vee r}}{r} \qquad \neg r}{\square}$$

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the clause set $\mathcal{C}$ is refutable, hence the CNF represented is unsatisfiable

$$(q \vee r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee r) \wedge \neg r$$

Many-Valued Propositional Logics

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why do we have only two truth values?

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1	0

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three-valued logic is employed in SQL to handle unknown values

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1. *(finite- or infinite-valued) Lukasiewicz logic is finitely axiomatisable, that is, there exists a finite sound and complete proof system*
2. *validity for Lukasiewicz logic is decidable (it is* coNP*-complete)*

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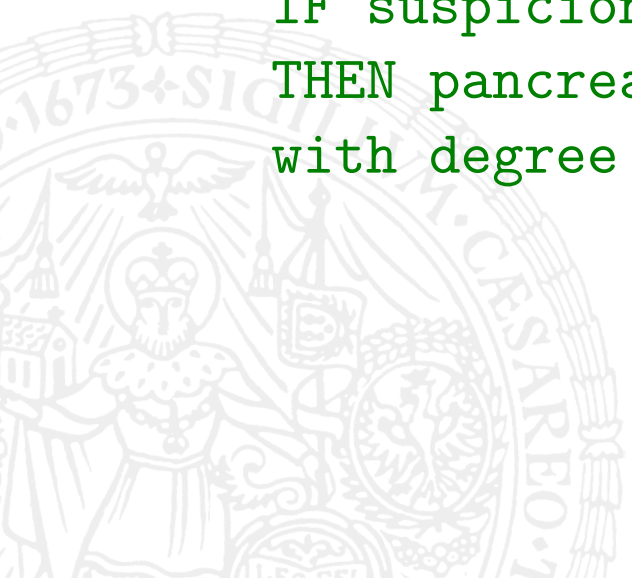

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- inference system of CADIAG-2 can be expressed as a infinite valued fuzzy logics
- representation showed inconsistencies in CADIAG-2