

Automated Reasoning

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Winter 2013



Happy New Year!



Summary Last Lecture

Definition

- a literal L is **maximal** if \exists ground σ such that for no other literal M : $M\sigma \succ_L L\sigma$
- L is **strictly maximal** if \exists ground σ such that for no other literal M : $M\sigma \succcurlyeq_L L\sigma$; here \succcurlyeq_L denotes the reflexive closure

Definition

ordered resolution

$$\frac{C \vee A \quad D \vee \neg B}{(C \vee D)\sigma}$$

ordered factoring

$$\frac{C \vee A \vee B}{(C \vee A)\sigma}$$

- 1 σ is a mgu of the atomic formulas A and B
- 2 $A\sigma$ is **strictly maximal** with respect to $C\sigma$; $\neg B\sigma$ is **maximal** with respect to $D\sigma$

Definition

subsumption and resolution can be combined in the following ways

1 forward subsumption

newly derived clauses subsumed by existing clauses are deleted

2 backward subsumption

existing clauses C subsumed by newly derived clauses D become inactive

inactive clauses are reactivated, if D is no ancestor of current clause

3 replacement

the set of all clauses (derived and initial) are frequently reduced under subsumption

Theorem

(ordered) resolution is complete under forward subsumption and tautology elimination

Outline of the Lecture

Early Approaches in Automated Reasoning

short recollection of Herbrand's theorem, Gilmore's prover, method of Davis and Putnam

Starting Points

resolution, tableau provers, Skolemisation, ordered resolution, redundancy and deletion

Automated Reasoning with Equality

paramodulation, ordered completion and proof orders, superposition

Applications of Automated Reasoning

Neuman-Stubblebinde Key Exchange Protocol, group theory, resolution and paramodulation as decision procedure, ...

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Paramodulation Calculus

Definition

- let \square be a fresh constant; let \mathcal{L} be our basic language
- terms of $\mathcal{L} \cup \{\square\}$ such that \square occurs exactly once, are called **contexts**
- empty context is denoted as \square
- for context $C[\square]$ and a term t
we write $C[t]$ for the replacement of \square by t



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we write $C[t]$ for the replacement of \square by t

Example

- let $\mathcal{L} = \{c, f, P\}$
- $P(f(\square)) =: C[\square]$ is a context
- $C[f(c)] = P(f(f(c)))$

Definition

$$\frac{C \vee A \quad D \vee \neg B}{(C \vee D)\sigma_1}$$

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- σ_1 is a mgu of A and B (A, B atomic)



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- σ_2 is a mgu of s and s'



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Example

consider $\mathcal{C} = \{c \neq d, b = d, a \neq d \vee a = c, a = b \vee a = d\}$

$$\frac{\frac{\frac{b = d \quad a = b \vee a = d}{a = d \vee a = d}}{a = d} \quad c \neq d}{a \neq c}$$

$$\frac{a = d \quad a \neq d \vee a = c}{d \neq d \vee a = c}$$

$$\frac{a = c}{a = c}$$

□

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- define the **paramodulation operator** $\text{Res}_p(\mathcal{C})$ as follows:

$$\text{Res}_p(\mathcal{C}) = \{D \mid D \text{ is paramodulant, etc. with premises in } \mathcal{C}\}$$



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Theorem

paramodulation is sound and complete: if F is a sentence and \mathcal{C} its clause form, then F is unsatisfiable iff $\Box \in \text{Res}_P^(\mathcal{C})$*



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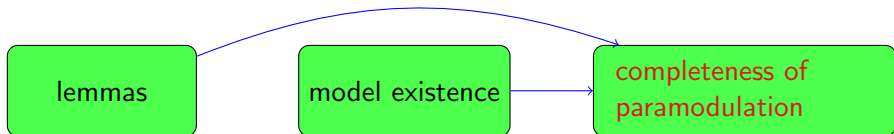
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Proof Plan.



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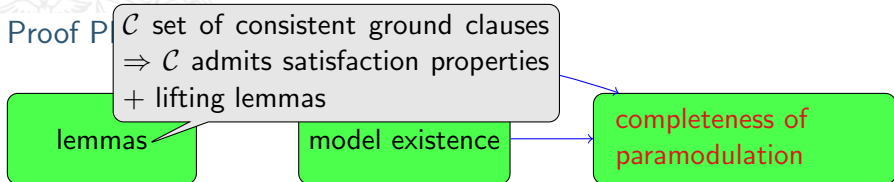
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A Problem with Lifting

Claim

- let τ_1 and τ_2 be a ground and consider

$$\frac{C_{\tau_1} \vee (s = t)_{\tau_1} \quad D_{\tau_2} \vee L_{\tau_2}[s'_{\tau_2}]}{C_{\tau_1} \vee D_{\tau_2} \vee L_{\tau_2}[t_{\tau_2}]}$$

where $s_{\tau_1} = s'_{\tau_2}$

- \exists mgu σ of s and s' , such that σ is more general then τ_1 and τ_2 and the following paramodulation step is valid

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Fact

observe that paramodulation *into* variables is allowed

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- we add the **functional reflexivity equation** $f(x) = f(x)$ from which we get $f(a) = f(b)$ by paramodulation **into a variable**
- then lifting becomes possible (using two steps)

$$\frac{a = b \quad f(x) = f(x)}{f(a) = f(b)} \quad \frac{f(a) = f(b) \quad f(x) = c}{f(f(b)) = c}$$

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Lemma

- let τ_1 and τ_2 be a ground and consider

$$\frac{C_{\tau_1} \vee (s = t)_{\tau_1} \quad D_{\tau_2} \vee L_{\tau_2}[x_{\tau_2}]}{C_{\tau_1} \vee D_{\tau_2} \vee L_{\tau_2}[f(t_{\tau_1})]}$$

where $x_{\tau_2} = f(s'_{\tau_3})$ and $s_{\tau_1} = s'_{\tau_3}$

- then the following paramodulation step is valid, trivially more general than the ground step

$$\frac{\frac{C \vee s = t \quad f(x) = f(x)}{C \vee f(s) = f(t)} \quad D \vee L[x]}{C \vee D \vee L[f(t)]}$$

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paramodulation is sound and complete: if F is a sentence and \mathcal{C} its clause form (containing all functional reflexive equations), then F is unsatisfiable iff $\Box \in \text{Res}_p^(\mathcal{C})$*

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Discussion

- alternative completeness proof employs an adaption of the semantic tree argument
- paramodulation is inefficient
- one idea to reduce the search space is to combine ordered resolution with paramodulation: ordered paramodulation

Ordered Paramodulation Calculus

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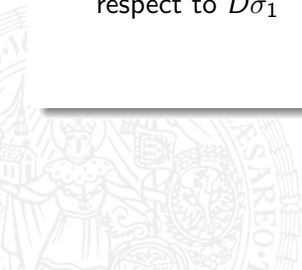
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Theorem

ordered paramodulation is sound and complete

Example

re-consider \mathcal{C}

$$c \neq d \quad b = d \quad a \neq d \vee a = c \quad a = b \vee a = d$$

together with the literal order:

$$\begin{aligned} a \neq b \succ a = b \succ a \neq c \succ a = c \succ a \neq d \succ a = d \\ \succ b \neq d \succ b = d \succ c \neq d \succ c = d \end{aligned}$$

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the following derivation is no longer admissible

$$\frac{\frac{\frac{b = d \quad a = b \vee a = d}{a = d \vee a = d}}{a = d} \quad c \neq d}{a \neq c} \quad \frac{\frac{\frac{\Pi}{a = d} \quad a \neq d \vee a = c}{d \neq d \vee a = c}}{a = c}}{\square}$$

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Example (cont'd)

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 }{c \neq d} \quad \square$$

Discussion

- ordered paramodulation is still too inefficient
- various refinements have been introduced, one is the superposition calculus

Employ Rewriting Techniques

Definitions

- **rewrite relation** ...



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- **rewrite relation** ...
- **normal form** ...



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Employ Rewriting Techniques

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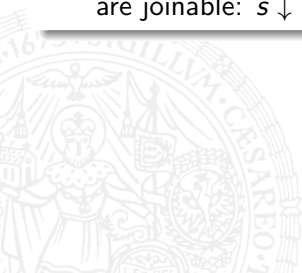
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- an equation $s = t$ **converges** (or has a **rewrite proof**) in \mathcal{R} if s and t are joinable: $s \downarrow t$



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Facts

- 1 a convergent (confluent & terminating) TRS forms a **decision procedure** for the underlying equational theory: $s \leftrightarrow^* t$ iff $s \downarrow t$

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Facts

- 1 a convergent (confluent & terminating) TRS forms a **decision procedure** for the underlying equational theory: $s \leftrightarrow^* t$ iff $s \downarrow t$
- 2 normalisation in a convergent TRS amounts to a **don't care nondeterminism**

Completion

Definition (superposition of rewrite rules)

$$\frac{s \rightarrow t \quad w[u] \rightarrow v}{(w[t] = v)\sigma}$$

σ mgu of s and u and u not a variable; then $(w[t] = v)\sigma$ is a critical pair



Completion

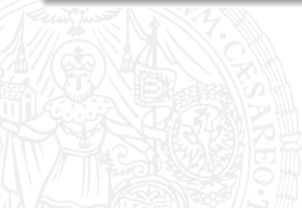
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Example

LPO is not total; x, y, u, v variables:

$$f(x, y) \not\prec_{\text{lpo}} f(u, w) \quad f(u, w) \not\prec_{\text{lpo}} f(x, y)$$

Ordered Rewriting

Definitions

- reduction orders that are total on **ground terms** are called **complete**
- \succ a reduction order; \mathcal{E} a set of equations; consider

$$\mathcal{E}^\succ = \{s\sigma \rightarrow t\sigma \mid s = t \in \mathcal{E}, s\sigma \succ t\sigma\}$$

- rules in \mathcal{E}^\succ are called **reductive instances** of equations in \mathcal{E}
- rewrite relation $\rightarrow_{\mathcal{E}^\succ}$ represents **ordered rewriting**



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- let \succ_{lpo} be a LPO induced by the precedence $+ \succ a \succ b \succ c$
- $b + c \succ_{lpo} c + b \succ_{lpo} c$
- commutativity $x + y = y + x$ yields the ordered rewrite derivation:

$$(b + c) + c \rightarrow (c + b) + c \rightarrow c + (c + b)$$

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equations \mathcal{E} are **ground convergent wrt** \succ if \mathcal{E}^\succ is ground convergent



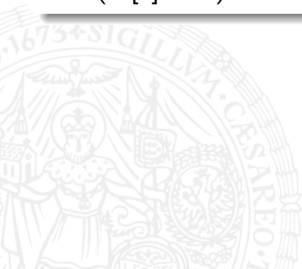
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Theorem

\succ a complete reduction order; a set of equations E is ground convergent wrt \succ iff \forall ordered critical pairs $(w[t] = v)\sigma$ (with overlapping term $w[u]\sigma$) and \forall ground substitutions τ : if $w[u]\sigma\tau \succ w[t]\sigma\tau$ and $w[u]\sigma\tau \succ v\sigma\tau$ then $w[t]\sigma\tau \downarrow v\sigma\tau$

Ordered Completion

deduction

$$\mathcal{E}; \mathcal{R} \vdash \mathcal{E} \cup \{s = t\}; \mathcal{R}$$

if $s \leftrightarrow_{\mathcal{E} \cup \mathcal{R}} w \leftrightarrow_{\mathcal{E} \cup \mathcal{R}} t, s \not\leq w, t \not\leq w$



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orientation

$$\mathcal{E} \cup \{s = t\}; \mathcal{R} \vdash \mathcal{E}; \mathcal{R} \cup \{s \rightarrow t\}$$

if $s \succ t$



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Ordered Completion

deduction	$\mathcal{E}; \mathcal{R} \vdash \mathcal{E} \cup \{s = t\}; \mathcal{R}$ if $s \leftrightarrow_{\mathcal{E} \cup \mathcal{R}} w \leftrightarrow_{\mathcal{E} \cup \mathcal{R}} t, s \not\leq w, t \not\leq w$	
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Definition

- a **proof** of $s = t$ wrt $\mathcal{E}; \mathcal{R}$ is

$$s = s_0 \rho_0 s_1 \rho_1 s_2 \cdots s_{n-1} \rho_{n-1} s_n = t \quad n \geq 0$$

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Fact

- 1 \exists *rewrite proof* iff the equations converge wrt $\mathcal{R} \cup \mathcal{E}^\succ$
- 2 whenever $\mathcal{E}; \mathcal{R} \vdash \mathcal{E}'; \mathcal{R}'$ then the same equations are provable in $\mathcal{E}; \mathcal{R}$ as in $\mathcal{E}'; \mathcal{R}'$; however proofs may become **simpler**