

Automated Reasoning

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Summary of Last Lecture

Definition

$$\frac{C \vee A \quad D \vee \neg B}{(C \vee D)\sigma} \text{ ORe}$$

$$\frac{C \vee s = t \quad D \vee \neg A[s']}{(C \vee D \vee \neg A[t])\sigma} \text{ OPm(L)}$$

$$\frac{C \vee s = t \quad D \vee u[s'] \neq v}{(C \vee D \vee u[t] \neq v)\sigma} \text{ SpL}$$

$$\frac{C \vee s \neq t}{C\sigma} \text{ ERR}$$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \text{ OFc}$$

$$\frac{C \vee s = t \quad D \vee A[s']}{(C \vee D \vee A[t])\sigma} \text{ OPm(R)}$$

$$\frac{C \vee s = t \quad D \vee u[s'] = v}{(C \vee D \vee u[t] = v)\sigma} \text{ SpR}$$

$$\frac{C \vee u = v \vee s = t}{(C \vee v \neq t \vee u = t)\sigma} \text{ EFc}$$

- ORe and OFc are **ordered resolution** and **ordered factoring**
- OPm(L), OPm(R), SpL, SpR stands for **ordered paramodulation** and **superposition** (left or right)
- ERR means **equality resolution** and EFc means **equality factoring**

Example

re-consider $\mathcal{C} = \{c \neq d, b = d, a \neq d \vee a = c, a = b \vee a = d\}$ together with the term order: $a \succ b \succ c \succ d$; without equality factoring only the following clause is derivable:

$$a \neq d \vee b = c \vee a = d$$

here the atom order is the multiset extension of \succ : $a = b \equiv \{a, b\} \succ \{a, d\} \equiv a = d$ and the literal order \succ_L is the multiset extension of the atom order: $a = c \succ_L a \neq d$

Lemma

non-redundant superposition inferences are liftable

Theorem

superposition is sound and complete; let F be a sentence and \mathcal{C} its clause form; then F is unsatisfiable iff $\Box \in \text{Res}_{\text{SP}}^(\mathcal{C})$*

Outline of the Lecture

Early Approaches in Automated Reasoning

short recollection of Herbrand's theorem, Gilmore's prover, method of Davis and Putnam

Starting Points

resolution, tableau provers, Skolemisation, ordered resolution, redundancy and deletion

Automated Reasoning with Equality

paramodulation, ordered completion and proof orders, superposition

Applications of Automated Reasoning

Neuman-Stubblebine Key Exchange Protocol, group theory Robbin's problem

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Neuman-Stubblebine Key Exchange Protocol, group theory Robbin's problem

Application ①: Issues of Security



Neuman-Stubblebine Key Exchange Protocol

Description

- Neuman-Stubblebine key exchange protocol aims to establish a secure key between two agents that already share secure keys with a trusted third party
- principals: Alice, Bob, Server



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- principals: Alice, Bob, Server

Conventions

| | |
|--|------------------------------|
| A, B, T identifiers of Alice, Bob, Server | K_{at} key between A and T |
| N_a, N_b nonce created by Alice, Bob | K_{bt} key between B and T |
| Time time span of key K_{ab} | K_{ab} key between A and B |
| $E_{key}(message)$ encryption of <i>message</i> using <i>key</i> | |

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Definition

we write

$A \longrightarrow B: M$ **Alice** sends **Bob** message *M*

The Protocol

1 $A \longrightarrow B: A, N_a$

Alice sends to Bob

- her identifier
- a freshly generated nonce



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Alice sends to Bob

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2 $B \longrightarrow T: B, E_{K_{bt}}(A, N_a, \text{Time}), N_b$

Bob encrypts the triple (A, N_a, Time) and sends to Server

- his identity
- encryption of (A, N_a, Time)
- new nonce



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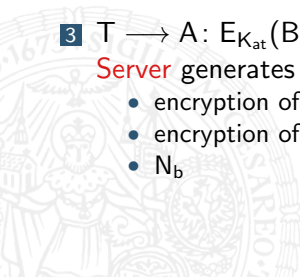
Bob encrypts the triple (A, N_a, Time) and sends to Server

- his identity
- encryption of (A, N_a, Time)
- new nonce

3 $T \longrightarrow A: E_{K_{at}}(B, N_a, K_{ab}, \text{Time}), E_{K_{bt}}(A, K_{ab}, \text{Time}), N_b$

Server generates K_{ab} and sends to Alice

- encryption of K_{ab} with key for Alice
- encryption of K_{ab} with key for Bob
- N_b



The Protocol

1 $A \longrightarrow B: A, N_a$

Alice sends to Bob

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- a freshly generated nonce

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Server generates K_{ab} and sends to Alice

- encryption of K_{ab} with key for Alice
- encryption of K_{ab} with key for Bob
- N_b

4 $A \longrightarrow B: E_{K_{bt}}(A, K_{ab}, \text{Time}), E_{K_{ab}}(N_b)$

Alice encrypts Bob's nonce with K_{ab} and forwards part of message

The Attack

Assumptions

- 1 intruder can intercept and record all sent messages
- 2 intruder can send messages and can forge the sender of a message
- 3 intruder can encrypt messages, when he finds out a key
- 4 intruder has no access to information private to Alice, Bob, or Server the server.
- 5 intruder cannot break any secure key



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- 1 $I(A) \longrightarrow B: A, N_a$

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- 3 $I(A) \longrightarrow B: E_{K_{bt}}(A, N_a, \text{Time}), E_{N_a}(N_b).$

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the problem is that keys and nonces can be confused

$$E_{K_{bt}}(A, K_{ab}, \text{Time}) \quad \text{and} \quad E_{K_{bt}}(A, N_a, \text{Time})$$

Formalisation in First-Order

Definition

definition of the language \mathcal{L} of the formalisation

Formalisation in First-Order

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1 individual constants: a, b, t, na, at, bt

- a, b, t are to be interpreted as the identifiers A, B , and T
- constant na refers to Alices's nonce
- at (bt) represents the key K_{at} (K_{bt})

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2 function constants: nb, tb, kt, key, sent, pair, triple, encr, quadr

- nb, tb, kt are unary; key, pair, encr are binary; sent, triple are ternary, and quadr is 4-ary
- nb, tb compute Bob's fresh nonce and the time-stamp Time
- kt computes of the new key
- the other constants act as containers as the formalisation is based on unary predicates

Definition (Definition (cont'd))

4 **predicate constants:** Ak , Bk , Tk , P , M , $Fresh$, $Nonce$, $Store_a$, $Store_b$

- Ak , Bk , Tk assert together with key existence of keys
- P represents principals
- M represents messages using the function sent
- $Fresh$ asserts that Bob is only interested in fresh nonces
- $Nonce$ denotes that its argument is a nonce
- $Store_a$, $Store_b$ denote information that is in the store of Alice or Bob



Definition (Definition (cont'd))

4 **predicate constants:** A_k , B_k , T_k , P , M , Fresh , Nonce , Store_a , Store_b

- A_k , B_k , T_k assert together with key existence of keys
- P represents principals
- M represents messages using the function sent
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Notation

we indicate the type of a bound variable in its name as subscript

the bound variable x_{na} indicates that this variable plays the role of the nonce N_a

Formalisation of Protocol

$A \longrightarrow B: A, N_a$

1: $Ak(\text{key}(at, t))$

2: $P(a)$

3: $M(\text{sent}(a, b, \text{pair}(a, na))) \wedge \text{Store}_a(\text{pair}(b, na))$



Formalisation of Protocol

$A \longrightarrow B: A, N_a$

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3: $M(\text{sent}(a, b, \text{pair}(a, na))) \wedge \text{Store}_a(\text{pair}(b, na))$

$B \longrightarrow T: B, E_{K_{bt}}(A, N_a, \text{Time}), N_b$

4: $Bk(\text{key}(bt, t))$

5: $P(b)$

6: $\text{Fresh}(na)$

7: $\forall x_a \ x_{na} (M(\text{sent}(x_a, b, \text{pair}(x_a, x_{na}))) \wedge \text{Fresh}(x_{na}) \rightarrow$
 $\rightarrow \text{Store}_b(\text{pair}(x_a, x_{na})) \wedge M(\text{sent}(b, t,$
 $\text{triple}(b, nb(x_{na}), \text{encr}(\text{triple}(x_a, x_{na}, \text{tb}(x_{na})), \text{bt}))))))$

$T \longrightarrow A: E_{K_{at}}(B, N_a, K_{ab}, \text{Time}), E_{K_{bt}}(A, K_{ab}, \text{Time}), N_b$

8: $\text{Tk}(\text{key}(\text{at}, a)) \wedge \text{Tk}(\text{key}(\text{bt}, b))$

9: $P(t)$

10: $\forall x_b \forall x_{nb} \forall x_a \forall x_{na} \forall x_{\text{time}} \forall x_{bt} \forall x_{at}$

$(M(\text{sent}(x_b, t, \text{triple}(x_b, x_{nb}, \text{encr}(\text{triple}(x_a, x_{na}, x_{\text{time}}), x_{bt})))) \wedge$
 $\wedge \text{Tk}(\text{key}(x_{at}, x_a)) \wedge \text{Tk}(\text{key}(x_{bt}, x_b)) \wedge \text{Nonce}(x_{na}) \rightarrow M(\text{sent}(t, x_a,$
 $\text{triple}(\text{encr}(\text{quadr}(x_b, x_{na}, \text{kt}(x_{na}), x_{\text{time}}), x_{at}),$
 $\text{encr}(\text{triple}(x_a, \text{kt}(x_{na}), x_{\text{time}}), x_{bt}), x_{nb}))))$

11: $\text{Nonce}(na)$

12: $\forall x \neg \text{Nonce}(\text{kt}(x))$

13: $\forall x (\text{Nonce}(\text{tb}(x)) \wedge \text{Nonce}(\text{nb}(x)))$



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8: $Tk(\text{key}(\text{at}, a)) \wedge Tk(\text{key}(\text{bt}, b))$

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Remark

formulas 11–13 are not part of the protocol, but prevents that the intruder can generate arbitrarily many keys

$A \longrightarrow B: E_{K_{bt}}(A, K_{ab}, \text{Time}), E_{K_{ab}}(N_b)$

14: $\forall x_{nb} \forall x_k \forall x_m \forall x_b \forall x_{na} \forall x_{time}$

$((M(\text{sent}(t, a, \text{triple}(\text{encr}(\text{quadr}(x_b, x_{na}, x_k, x_{time}), at), x_m, x_{nb})))) \wedge$
 $\wedge \text{Store}_a(\text{pair}(x_b, x_{na}))) \rightarrow$
 $\rightarrow M(\text{sent}(a, x_b, \text{pair}(x_m, \text{encr}(x_{nb}, x_k)))) \wedge Ak(\text{key}(x_k, x_b)))$

15: $\forall x_k \forall x_a \forall x_{na}$

$((M(\text{sent}(x_a, b, \text{pair}(\text{encr}(\text{triple}(x_a, x_k, \text{tb}(x_{na})), bt),$
 $\text{encr}(nb(x_{na}), x_k)))) \wedge$
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Fact

SPASS verifies that the protocol terminates in less than a millisecond

$\mathcal{G} \models \exists x (Ak(\text{key}(x, a)) \wedge Bk(\text{key}(x, b)))$

Formalisation of the Intruder

extend \mathcal{L} by predicate constants **lk** and **lm**

Behaviour of Intruder

$$16: \forall x_a x_b x_m (M(\text{sent}(x_a, x_b, x_m)) \rightarrow \text{lm}(x_m))$$

$$17: \forall u v (\text{lm}(\text{pair}(u, v)) \rightarrow \text{lm}(u) \wedge \text{lm}(v))$$

\vdots

$$20: \forall u v (\text{lm}(u) \wedge \text{lm}(v) \rightarrow \text{lm}(\text{pair}(u, v)))$$

\vdots

$$23: \forall x y u ((P(x) \wedge P(y) \wedge \text{lm}(u)) \rightarrow M(\text{sent}(x, y, u)))$$

$$24: \forall u v ((\text{lm}(u) \wedge P(v)) \rightarrow \text{lk}(\text{key}(u, v)))$$

$$25: \forall u v w ((\text{lm}(u) \wedge \text{lk}(\text{key}(v, w) \wedge P(w)) \rightarrow \text{lm}(\text{encr}(u, v)))$$

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SPASS shows that the protocol insecure in less than a millisecond

$$\mathcal{H} \models \exists x (\text{lk}(\text{key}(x, b)) \wedge \text{Bk}(\text{key}(x, a)))$$

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Fact

\mathcal{H} extends \mathcal{G} by 16–25

SPASS shows that the protocol insecure in less than a millisecond

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Application ②: Robbin's Problem



Definition

$\mathcal{B} = \langle B; +, \cdot, \bar{}, 0, 1 \rangle$ is a **Boolean algebra** if

1 $\langle B; +, 0 \rangle$ and $\langle B; \cdot, 1 \rangle$ are commutative monoids

2 $\forall a, b, c \in B$:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad a + (b \cdot c) = (a + b) \cdot (a + c)$$

3 $\forall a \in B$: $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$

\bar{a} is called **complement** (or **negation**) of a



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Definition

consider the following axioms:

$$x + y = y + x$$

commutativity

$$(x + y) + z = x + (y + z)$$

associativity

$$n(n(x) + y) + n(n(x) + n(y)) = x$$

Huntington equation

the operation $n(\cdot)$ is just **complement**

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Definition

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associativity

$$\overline{\bar{x} + \bar{y}} + \overline{\bar{x} + \bar{y}} = x$$

Huntington equation

the operation $n(\cdot)$ is just **complement**

Theorem

*the provided axioms form a minimal axiomatisation of Boolean algebras,
that is all axioms are independent from each other*



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Example

recall $x \cdot y = \overline{\overline{x} + \overline{y}}$, thus

$$\overline{\overline{x} + y} + \overline{\overline{x} + \overline{y}} = x \cdot \overline{y} + x \cdot y = x \cdot (\overline{y} + y) = x$$



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Definition

Robbins equation:

$$\overline{\overline{x + y} + \overline{x + y}} = x \tag{R}$$

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Definition

Robbins equation:

$$\overline{\overline{x + y} + \overline{x + \overline{y}}} = x \quad (\text{R})$$

Example

$$\overline{\overline{x + y} + \overline{x + \overline{y}}} = (x + y) \cdot (x + \overline{y}) = x + (y\overline{y}) = x$$

Robbins Question

Question ①

Does Huntington's equation follow from (i) commutativity (ii) associativity and (iii) Robbins equation?



Robbins Question

Question ①

Does Huntington's equation follow from (i) commutativity (ii) associativity and (iii) Robbins equation?

Answer

McCune (or better EQP) says **yes**



Robbins Question

Question ①

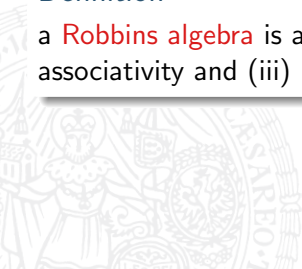
Does Huntington's equation follow from (i) commutativity (ii) associativity and (iii) Robbins equation?

Answer

McCune (or better EQP) says **yes**

Definition

a **Robbins algebra** is an algebra satisfying (i) commutativity (ii) associativity and (iii) Robbins equation



Robbins Question

Question ①

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Question ②

Is any Robbins algebra a Boolean algebra?

Auxiliary Lemmas

Lemma

a Robbins algebra satisfying $\exists x(x + x = x)$ is a Boolean algebra

Proof (Sketch).

automatically provable by EQP in about 5 seconds



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Lemma

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Proof (Sketch).

- 1 originally the Lemma was manually proven by Steve Winker
- 2 based on the above lemma EQP can find a proof in about 40 minutes

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a Robbins algebra satisfying $\exists x \exists y (\overline{x + y} = \bar{x})$ is a Boolean algebra

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Theorem

commutativity, associativity, and Robbins equation minimally axiomatise Boolean algebra

Proof (of last lemma).

| | |
|--|------------------------------|
| $n(n(n(x) + y) + n(x + y)) = y$ | 7, (R) |
| $n(n(n(x + y) + n(x) + y) + y) = n(x + y)$ | 10, [7 \rightarrow 7] |
| $n(n(n(n(x) + y) + x + y) + y) = n(n(x) + y)$ | 11, [7 \rightarrow 7] |
| $n(n(n(n(x) + y) + x + 2y) + n(n(x) + y)) = y$ | 29, [11 \rightarrow 7] |
| $n(n(n(n(n(x) + y) + x + 2y) + n(n(x) + y) + z) +$ $+ n(y + z)) = z$ | 54, [29 \rightarrow 7] |
| $n(n(n(n(n(x) + y) + x + 2y) + n(n(x) + y) +$ $+ n(y + z) + z) + z) = n(y + z)$ | 217, [54 \rightarrow 7] |
| $n(n(n(n(n(n(x) + y) + x + 2y) + n(n(x) + y) +$ $+ n(y + z) + z) + z + u) + n(n(y + z) + u)) = u$ | 674, [217 \rightarrow 7] |
| $n(n(n(n(3x) + x) + n(3x)) + n(n(n(3x) + x) + 5x)) =$ $= n(n(3x) + x)$ | 6736, [10 \rightarrow 674] |

Proof.

$$n(n(n(3x) + x) + 5x) = n(3x) \quad 8855, [6736 \rightarrow 7]$$

$$n(n(n(n(3x) + x) + n(3x) + 2x)) = n(n(3x) + x) + 2x \quad 8865, [8855 \rightarrow 7]$$

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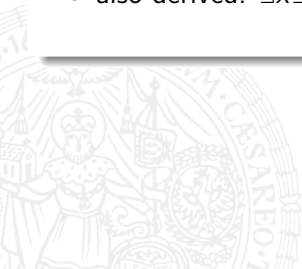
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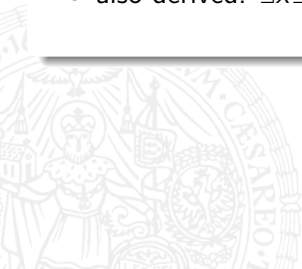
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Remarks

- SPASS could not find proof in 12 hours
- mkbtt cannot parse the problem ☺

Equational Prover EQP

Definition

- EQP is restricted to equational logic and performs AC unification and matching
- based on basic superposition, that is, paramodulation into substitution parts of terms are forbidden
- incomplete heuristics



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Definition

- AC unifiers are found by finding a **basis** of a linear Diophantine equation
- the complete set of unifiers is given as linear combinations of (members of) the basis

Definition

- a subset yields **potential unifier** if all unification conditions except unification of subterms are fulfilled
- the **super-0 strategy** restricts the number of AC unifiers by ignoring supersets if a potential unifier is found



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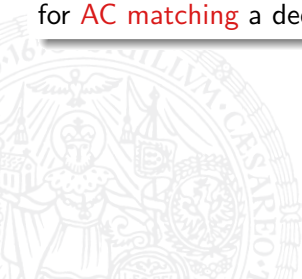
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Definitions

- the **weight** of a pair of equations be the sum of the size of its members
- the **age** of a pair is the sum of the ages of its members

Definition

a **pairing algorithm** used to select the next equation:

- 1 either the lightest or the oldest pair (not yet selected) is chosen
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- subsequent experiments searched for shorter proofs
- yielded direct proof without the use of Winker's lemmas

Thank You for Your Attention!

