Automated Reasoning

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Summary Last Lecture

Selection of Applications

- Program Analysis; logical product of abstract interpreters
- Databases; disjunctive datalog
- Programming Languages; types as formulas
- Computational Complexity; implicit complexity

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Lessons Learnt

- (mathematical) logic is the science of (mathematical) reasoning
- logic has been and is very successfully used as workbench for various areas in computer science
- applications are not trivial (in both senses)

Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, Löwenheim-Skolem, compactness, model existence theorem, natural deduction, completeness, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

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a first-order language is determined by specifying its constants, variables, logical symbols, auxiliary (brackets, comma)

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Definition

- individual constants: $k_0, k_1, \ldots, k_j, \ldots$ denoted c, d, etc.
- function constants with i arguments: $f_0^i, f_1^i, \ldots$ denoted f, g, h etc.
- predicate constants with i arguments: $R_0^i, R_1^i, \ldots$ denoted as P, Q, R, etc.

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- variables: $x_0, x_1, \dots, x_j, \dots$ denoted x, y, z, etc.

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Example

the language of arithmetic $\mathcal{L}_{\mathsf{arith}}$ contains $=$ and consists of

- individual constant 0
- function constants s, $+$, $\cdot$
- predicate constant $<$

Terms of a Language

Definition

terms (of $\mathcal{L}$) are defined as follows

- any individual constant c in $\mathcal{L}$ is a term
- any variable x is a term
- if $t_1, \ldots, t_n$ are terms, f an n-ary function constant in $\mathcal{L}$, then $f(t_1, \ldots, t_n)$ is a term

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Convention

if the language $\mathcal{L}$ is clear from context the phrase "of $\mathcal{L}$" will be dropped

Formulas (of a Language)

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- $P(t_1, \ldots, t_n)$ is an atomic formula; P a constant of arity n, t_i terms
- $t_1 = t_2$ is an atomic formula, if $=$ is present

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- atomic formulas are formulas
- A and B are frms: $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ are formulas
- if A a formula, x a variable, then

  $$\forall x A \qquad \exists x A$$

  are formulas

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if brackets are not necessary they are omitted:

$$\exists, \forall \;>\; \neg \;>\; \vee, \wedge \;>\; \rightarrow \qquad \text{right-associativity of } \rightarrow$$

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consider $\mathcal{L}_{\text{arith}}$, which of the following are formulas over $\mathcal{L}_{\text{arith}}$?

- $x < y \wedge \neg \exists z (x < z \wedge z < y)$
- $\forall x (x = 0) \rightarrow \exists x (x = 0)$
- $\forall x (x < y \wedge \exists x (y = x))$

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we write $c^{\mathcal{A}}$. $f^{\mathcal{A}}$, and $P^{\mathcal{A}}$, instead of $a(c)$, $a(f)$, and $a(P)$; for brevity we write $=$ for the equality sign and the identity relation

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Remark

a structure is equivalent to the definition of model in LICS

Definition

- an environment for $\mathcal{A}$ is a mapping $\ell \colon \{x_n \mid n \in \mathbb{N}\} \to A$
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Definition

the value of a term t (wrt interpretation $\mathcal{I}$)

$$t^{\mathcal{I}} = \begin{cases} \ell(t) & \text{if } t \text{ a variable} \\ c^{\mathcal{A}} & \text{if } t = c \\ f^{\mathcal{A}}(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}}) & \text{if } t = f(t_1, \ldots, t_n),\ n \geqslant 1 \end{cases}$$

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$F, G_1, \ldots, G_n$ be formulas

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- $\mathcal{N} = (\mathbb{N}, 0, \mathrm{succ}, +, \cdot, <)$ denote the standard structure of arithmetic

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consider the formula $A := x < y \land \neg\exists z(x < z \land z < y)$

- $\mathcal{N} = (\mathbb{N}, 0, \mathrm{succ}, +, \cdot, <)$ denote the standard structure of arithmetic
- $\ell(x) = 1, \ell(y) = 2$

Definition

F, $G_1, \dots, G_n$ be formulas

- $G_1, \dots, G_n \models F$ iff
  $\forall$ interpretations $\mathcal{I}$ of all $G_1, \dots, G_n$ such that
  $\mathcal{I}$ models $G_1, \dots, G_n$, we have $\mathcal{I}$ models F
- F is called satisfiable
  if $\exists$ an interpretation that is a model of F $\quad$ $\mathsf{Sat}(F)$
- F is valid if
  F is satisfiable in any interpretation $\quad$ $\models F$

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then $(\mathcal{N}, \ell) \models A$

Example

consider concatenation of lists

$$\mathsf{app}(\mathsf{nil}, Ys) = Ys \qquad \mathsf{app}(Xs, Ys) = \mathsf{cons}(\mathsf{head}(Xs), \mathsf{app}(\mathsf{tail}(Xs), Ys))$$

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and the language $\mathcal{L}$:

$$\mathrm{nil},\ \ \mathrm{head}(Xs),\ \ \mathrm{tail}(Xs),\ \ \mathrm{cons}(X, Xs),\ \ =,\ \ \text{and}\ \ \mathrm{App}(Xs, Ys, Zs)$$

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$$\forall x\ \mathrm{App}(\mathrm{nil}, x, x) \wedge$$
$$\wedge\ \forall x \forall y \forall z\, (x \neq \mathrm{nil} \wedge \mathrm{App}(\mathrm{tail}(x), y, z) \rightarrow \mathrm{App}(x, y, \mathrm{cons}(\mathrm{head}(x), z)))$$

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Example

define

$$I_n := \forall x_1 \ldots \forall x_{n-1} \exists y (x_1 \neq y \wedge \cdots \wedge x_{n-1} \neq y)$$

if $\mathcal{I} \models I_n$, then $\mathcal{I}$ has at least n elements

Definition

let F be a formula such that x occurs in F

- x is bound if it occurs inside the scope of a quantifier
- otherwise x is free
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consider $\forall x(\mathrm{P}(x) \wedge \mathrm{Q}(x, y))$; then x is bound and y is free

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consider $\forall x(\mathsf{P}(x) \wedge \mathsf{Q}(x, y))$; then x is bound and y is free

Notation

let F be a formula, x a free variable in F, t a term

- we sometimes write $F(x)$ instead of F to indicate x
- $F(t)$ denotes the replacement of x by t
- $F(t)$ is an instance of $F(x)$

Definition

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$\forall$ *formulas F and all sets of formulas $\mathcal{G}$: $\mathcal{G} \models F$ iff* $\neg\,\mathsf{Sat}(\mathcal{G} \cup \{\neg F\})$

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Lemma

1. *let $\mathcal{I}_1 = (\mathcal{A}_1, \ell_1)$ and $\mathcal{I}_2 = (\mathcal{A}_2, \ell_2)$ be interpretations*
2. *the universes of $\mathcal{I}_1$, $\mathcal{I}_2$ coincide*
3. *$\mathcal{I}_1$ and $\mathcal{I}_2$ coincide on the constants and variables occurring in F*

then $\mathcal{I}_1 \models F$ iff $\mathcal{I}_2 \models F$

Toy Example: Logic as Modelling Language

Argument ①

1. a mother or father of a person is an ancestor of that person
2. an ancestor of an ancestor of a person is an ancestor of a person
3. Sarah is the mother of Isaac, Isaac is the father of Jacob
4. Thus, Sarah is an ancestor of Jacob

Argument ②

1. a square or cube of a number is a power of that number
2. a power of a power of a number is a power of that number
3. 64 is the cube of 4, four is the square of 2
4. Thus, 64 is a power of 2

Toy Example: Logic as Modelling Language

Argument ①

1. $\mathsf{M}(x, y) \vee \mathsf{F}(x, y) \to \mathsf{A}(x, y)$
2. $\mathsf{A}(x, y) \wedge \mathsf{A}(y, z) \to \mathsf{A}(x, z)$
3. $\mathsf{M}(\mathsf{Sarah}, \mathsf{Isaac}) \wedge \mathsf{F}(\mathsf{Isaac}, \mathsf{Jacob})$
4. Thus $\mathsf{A}(\mathsf{Sarah}, \mathsf{Jacob})$

Argument ②

1. $\mathsf{S}(x, y) \vee \mathsf{C}(x, y) \to \mathsf{P}(x, y)$
2. $\mathsf{P}(x, y) \wedge \mathsf{P}(y, z) \to \mathsf{P}(x, z)$
3. $\mathsf{C}(64, 4) \wedge \mathsf{S}(4, 2)$
4. Thus $\mathsf{P}(64, 2)$

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1. $\mathsf{R}_1(x,y) \lor \mathsf{R}_2(x,y) \to \mathsf{R}_3(x,y)$
2. $\mathsf{R}_3(x,y) \land \mathsf{R}_3(y,z) \to \mathsf{R}_3(x,z)$
3. $\mathsf{R}_1(\mathsf{c}_1,\mathsf{c}_2) \land \mathsf{R}_2(\mathsf{c}_2,\mathsf{c}_3)$
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Argument ①$\models$ ②?

$$\left\{\begin{array}{c} R_1(x,y) \lor R_2(x,y) \to R_3(x,y) \\ R_3(x,y) \land R_3(y,z) \to R_3(x,z) \\ R_1(c_1,c_2) \land R_2(c_2,c_3) \end{array}\right\} \models R_3(c_1,c_3)$$

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Structure $\mathcal{A}$

$c_1^{\mathcal{A}}$	Sarah	64	$R_1^{\mathcal{A}}(x,y)$	x mother of y	x square of y
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Toy Example: Logic as Modelling Language

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Question

how to automate?

A Bit of History

Fact

the idea of automated reasoning is (very) old

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  a machine to "compute" whether a given argument is sound

we already know that a 'calculus ratiocinator' cannot exist

Undecidability of First-Order Logic

Theorem

1. *the decision problem for the consequence relation is undecidable*
2. *the set of valid first-order formulas is not recursive*

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- encoding of TMs as first-order formulas
- reduction from Post correspondence problem ∎

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Proof.

- the set of all formulas (over a countable language) is countable
- completeness yields that one can enumerate all valid formulas ■

Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, Löwenheim-Skolem, compactness, model existence theorem, natural deduction, completeness, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

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Lemma

let A, B be sets; $m\colon A \to B$ be a bijection; if $\mathcal{A}$ is a structure with domain A, then $\exists$ structure $\mathcal{B}$ with $\mathcal{A} \cong \mathcal{B}$

Isomorphism Theorem

let $\mathcal{A}$, $\mathcal{B}$ be structures such that $m \colon \mathcal{A} \cong \mathcal{B}$, then for all sentences F:
$\mathcal{A} \models F$ iff $\mathcal{B} \models F$

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1. let $\mathcal{I} = (\mathcal{A}, \ell)$, define $\ell^m = m \circ \ell$, set $\mathcal{J} = (\mathcal{B}, \ell^m)$

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   - base case $F = (s = t)$

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∎

Corollary

1. $\forall$ *formula* F *that has a finite model has a model in the domain* $\{0, 1, 2, \ldots, n\}$
2. $\forall$ *formula* F *that has a countable infinite model has a model whose domain is* $\mathbb{N}$

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combination of both lemmas ∎

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Example

consider $\mathcal{L} = \{\fallingdotseq\}$ and

$$E :\Longleftrightarrow \forall x\ x \fallingdotseq x \wedge \forall x \forall y\ (x \fallingdotseq y \wedge y \fallingdotseq x) \wedge$$
$$\forall x \forall y \forall z\ ((x \fallingdotseq y \wedge y \fallingdotseq z) \rightarrow x \fallingdotseq z)$$
$$F :\Longleftrightarrow \forall x \forall y\ x \fallingdotseq y$$

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$$\begin{aligned} E &:\Longleftrightarrow \forall x\; x \fallingdotseq x \land \forall x \forall y\; (x \fallingdotseq y \land y \fallingdotseq x) \land \\ &\qquad \forall x \forall y \forall z\; ((x \fallingdotseq y \land y \fallingdotseq z) \to x \fallingdotseq z) \\ F &:\Longleftrightarrow \forall x \forall y\; x \fallingdotseq y \end{aligned}$$

if $\mathcal{M}$ and $\mathbb{N}$ are countable infinite and $\mathcal{M} \models E \land F$, $\mathcal{N} \models E \land F$, then $\mathcal{M} \cong \mathcal{N}$

Compactness and Löwenheim-Skolem

Theorem (Compactness Theorem)

if every finite subset of a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a model

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Proof Idea.

employ compactness to show that $\mathcal{G}$ has an infinite model and Löwenheim-Skolem to show that this model is countable ■

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4. f is a surjective homomorphism, the proof of the isomorphism lemma holds ∎

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first-order logic features the following three theorems

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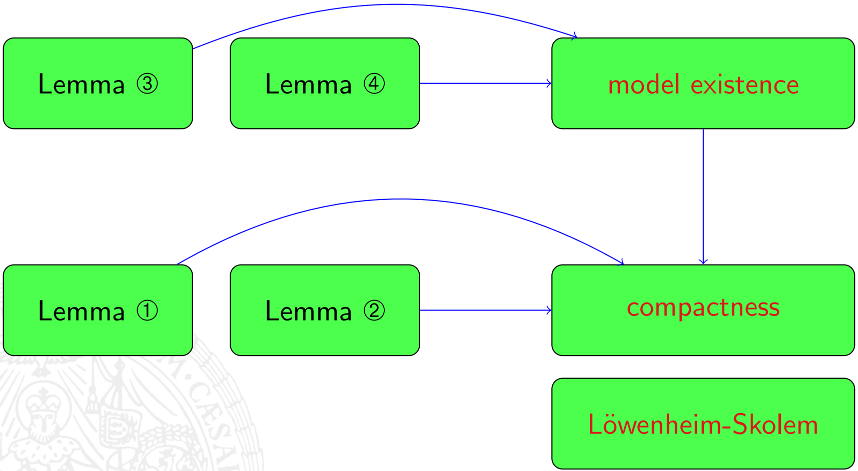
in proof, we restrict the logical symbols to $\neg$, $\vee$, $\exists$, and $=$

Howto Prove Compactness and Löwenheim-Skolem



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Definition

we call the properties (of S) in the lemma satisfaction properties

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Proof.

we treat the case of disjunction

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- wlog $\mathcal{G}_1 = \mathcal{G}_1' \cup \{E\}$, $\mathcal{G}_2 = \mathcal{G}_2' \cup \{F\}$, and $\mathcal{G}_1', \mathcal{G}_2' \subseteq \mathcal{G}$ finite

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- hence $\mathcal{G}_1' \cup \mathcal{G}_2' \cup \{E\} \in S$ or $\mathcal{G}_1' \cup \mathcal{G}_2' \cup \{F\} \in S$
- contradiction ∎

Compactness and Löwenheim-Skolem Theorem

$\mathcal{L}$ base language; $\mathcal{L}^+ \supseteq \mathcal{L}$ infinitely many new individual constants

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Theorem (Model Existence Theorem)

1. *if S^* is a set of formula sets of $\mathcal{L}^+$ having the satisfaction properties, then $\forall$ formula sets $\mathcal{G} \in S^*$ of $\mathcal{L}$, $\exists\ \mathcal{M}$, $\mathcal{M} \models \mathcal{G}$*
2. *$\forall$ elements m of $\mathcal{M}$: m denotes term in $\mathcal{L}^+$*

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Remark

the statement and the proof of the compactness theorem do not refer to provability; compactness is extensible to non-enumerable language

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Theorem (Löwenheim-Skolem Theorem)

if a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a countable model

Proof.

the model $\mathcal{M}$ constructed is countable ∎

How to Prove Completeness

model existence

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Lemma ③

Lemma ④

model existence

How to Prove Completeness

$\mathcal{G}$ has closure properties $\Rightarrow \exists$ model $\mathcal{M}$, $\mathcal{M} \models \mathcal{G}$

Lemma ③

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How to Prove Completeness

S admits satisfaction properties $\Rightarrow$
$\mathcal{G} \in S$ admits closure properties

Lemma ③

Lemma ④

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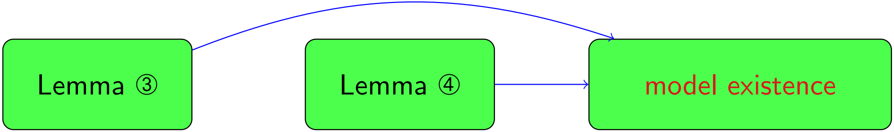
How to Prove Completeness

$$\begin{tikzcd}
\boxed{\text{Lemma ③}} \arrow[rr, bend left] & \boxed{\text{Lemma ④}} \arrow[r] & \boxed{\text{model existence}}
\end{tikzcd}$$

How to Prove Completeness



Definition

for any formal system; if $\neg\ \exists$ proof of $\perp$ from a formula set $\mathcal{G}$, we say $\mathcal{G}$ is consistent

How to Prove Completeness

$$\begin{tikzcd} \text{Lemma } \textcircled{3} \arrow[rr, bend left] & \text{Lemma } \textcircled{4} \arrow[r] & \text{model existence} \end{tikzcd}$$

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Lemma ⑥ | model existence | completeness

How to Prove Completeness

Lemma 3

Lemma 4 → model existence

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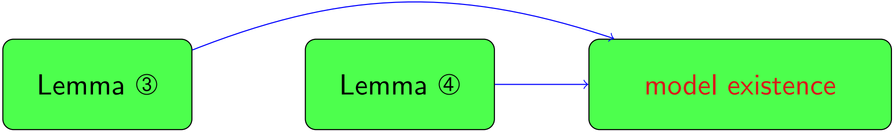
S set of consistent sets $\Rightarrow$ S admits satisfaction properties

Lemma 6

model existence

completeness

How to Prove Completeness



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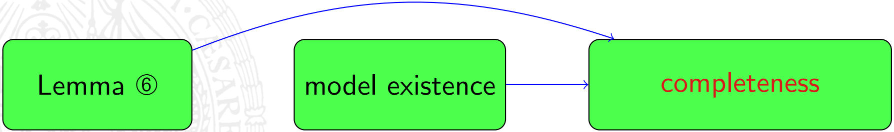

Later We Exploit the Proof

Lemma

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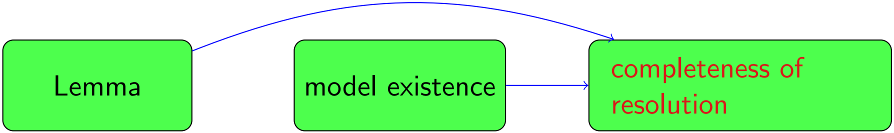

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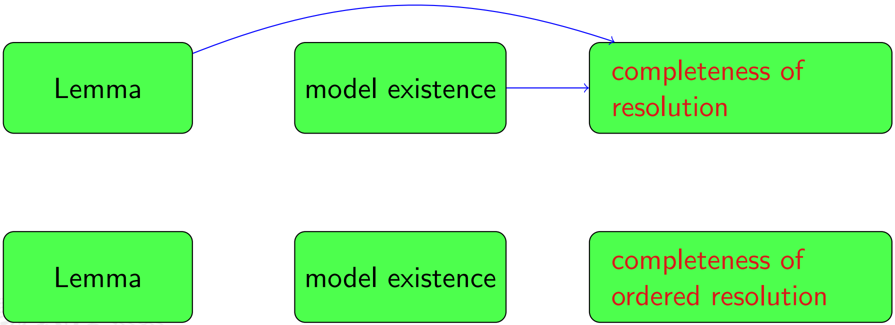

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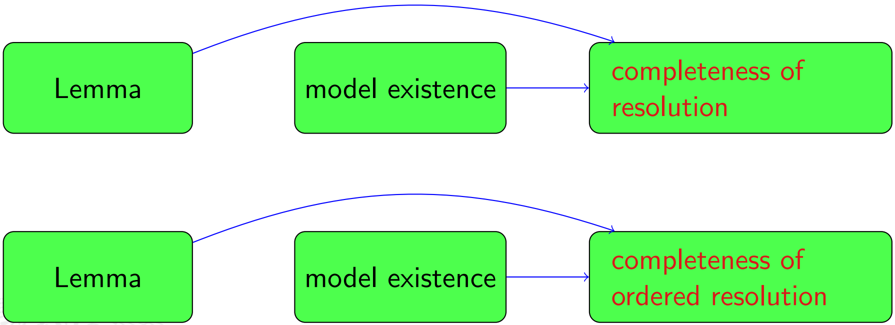

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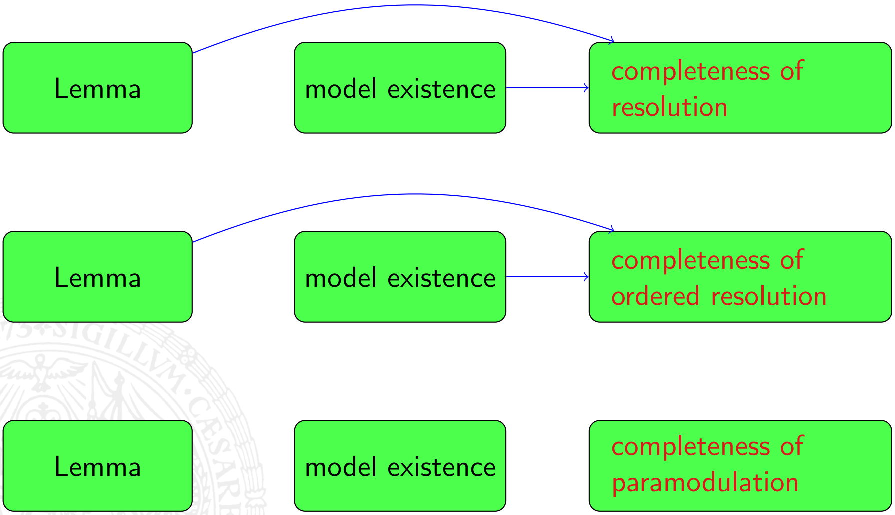

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