Automated Reasoning

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Summary Last Lecture

Compactness Theorem

if every finite subset of a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a model

Löwenheim-Skolem Theorem

if a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a countable model

compactness

Löwenheim-Skolem

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Lemma ①

Lemma ②

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model existence

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∃ satisfaction properties

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S^* admits satisfaction properties

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S admits satisfaction properties $\Rightarrow$ $\mathcal{G} \in S$ is satisfiable

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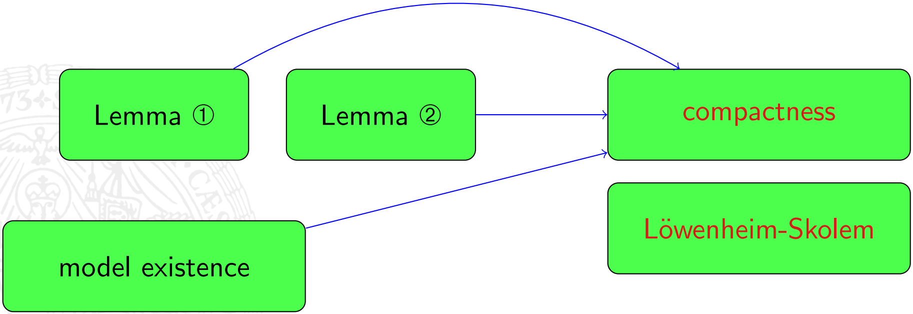

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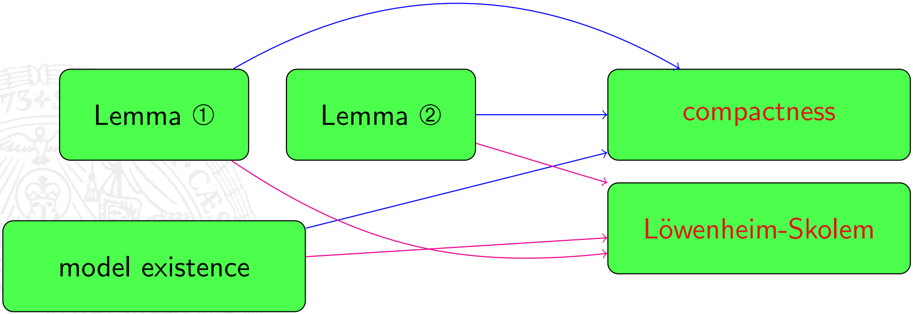

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Löwenheim-Skolem Theorem

if a set of formulas $\mathcal{G}$ has a countable model —

model existence

Lemma ①

let S be the set of satisfiable sets of formulas; pick $\mathcal{G} \in S$

1. *if $\mathcal{G}_0 \subseteq \mathcal{G}$, then $\mathcal{G}_0 \in S$*
2. *no formula F and $\neg F$ in $\mathcal{G}$*
3. *if $\neg\neg F \in \mathcal{G}$, then $\mathcal{G} \cup \{F\} \in S$*
4. *if $(E \vee F) \in \mathcal{G}$, then $\mathcal{G} \cup \{E\} \in S$ or $\mathcal{G} \cup \{F\} \in S$*
5. *if $\neg(E \vee F) \in \mathcal{G}$, then $\mathcal{G} \cup \{\neg E\} \in S$ and $\mathcal{G} \cup \{\neg F\} \in S$*
6. *if $\exists x F(x) \in \mathcal{G}$, the constant c doesn't occur in $\mathcal{G}$, then $\mathcal{G} \cup \{F(c)\} \in S$*
7. *if $\neg\exists x F(x) \in \mathcal{G}$, then $\forall$ terms t, $\mathcal{G} \cup \{\neg F(t)\} \in S$*
8. *for any term t, $\mathcal{G} \cup \{t = t\} \in S$*
9. *if $\{F(s), s = t\} \subseteq \mathcal{G}$, then $\mathcal{G} \cup \{F(t)\} \in S$*

Definition

we call the properties (of S) in the lemma satisfaction properties

Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, Löwenheim-Skolem, compactness, model existence theorem, completeness, natural deduction, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

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Model Existence

$\mathcal{L}$ base language; $\mathcal{L}^+ \supseteq \mathcal{L}$ infinitely many new individual constants

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Theorem (Model Existence Theorem)

1. *if S^* is a set of formula sets of $\mathcal{L}^+$ having the satisfaction properties, then $\forall$ formula sets $\mathcal{G} \in S^*$ of $\mathcal{L}$, $\exists$ $\mathcal{M}$, $\mathcal{M} \models \mathcal{G}$*
2. *$\forall$ elements m of $\mathcal{M}$: m denotes term in $\mathcal{L}^+$*

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2. *$\forall$ elements m of $\mathcal{M}$: m denotes term in $\mathcal{L}^+$*

$\mathcal{G}$ has closure properties $\Rightarrow \exists$ model $\mathcal{M}$, $\mathcal{M} \models \mathcal{G}$

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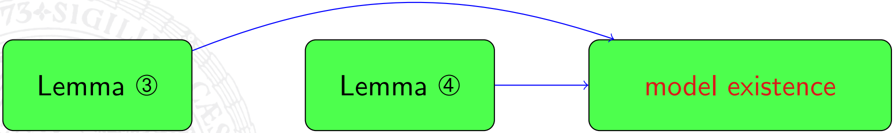

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3. *if $(E \vee F) \in \mathcal{G}$, then $E \in \mathcal{G}$ or $F \in \mathcal{G}$*

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5. *if $\exists x F(x) \in \mathcal{G}$, then $\exists$ term t (of $\mathcal{L}^+$), $F(t) \in \mathcal{G}$*

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7. $\forall$ term t (of $\mathcal{L}^+$), $t = t \in \mathcal{G}$

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8. *if $F(s) \in \mathcal{G}$, $s = t \in \mathcal{G}$, then $F(t) \in \mathcal{G}$*

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we call the properties of $\mathcal{G}$ closure properties

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Proof of Model Existence

by Lemma ④ and Lemma ③ ∎

Proof of Lemma ③

(no identity, no function symbols)

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Proof

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$$c^{\mathcal{M}} := c$$

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$$c^{\mathcal{M}} := c$$
3. $\forall$ predicate constant P, $\forall$ terms $t_1, \ldots, t_n$:
$$(t_1, \ldots, t_n) \in P^{\mathcal{M}} \iff P(t_1, \ldots, t_n) \in \mathcal{G}$$

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Proof (cont'd)

5. definition of $\mathcal{M}$ takes care of the demand that every element of its domain is the denotation of a term

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Proof (cont'd)

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- for the base case, let $F = P(t_1, \ldots, t_n)$, if $F \in \mathcal{G}$, then by definition $(t_1, \ldots, t_n) \in P^{\mathcal{M}}$; hence $\mathcal{M} \models F$

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  by induction hypothesis: $\mathcal{M} \models G(t)$

Proof (cont'd)

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Model Construction in a Picture

set of terms over $\mathcal{L}^+$

$$\begin{array}{cc}
 & \vdots \\
 & x_7 \\
\vdots & x_6 \\
k_5 & x_5 \\
k_4 & x_4 \\
k_3 & x_3 \\
k_2 & x_2 \\
k_1 & x_1 \\
k_0 & x_0
\end{array}$$

Model Construction in a Picture

set of terms over $\mathcal{L}^+$		domain of $\mathcal{M}$	
	$\vdots$		$\vdots$
	x_7		x_7
$\vdots$	x_6	$\vdots$	x_6
k_5	x_5	k_5	x_5
k_4	x_4	k_4	x_4
k_3	x_3	k_3	x_3
k_2	x_2	k_2	x_2
k_1	x_1	k_1	x_1
k_0	x_0	k_0	x_0

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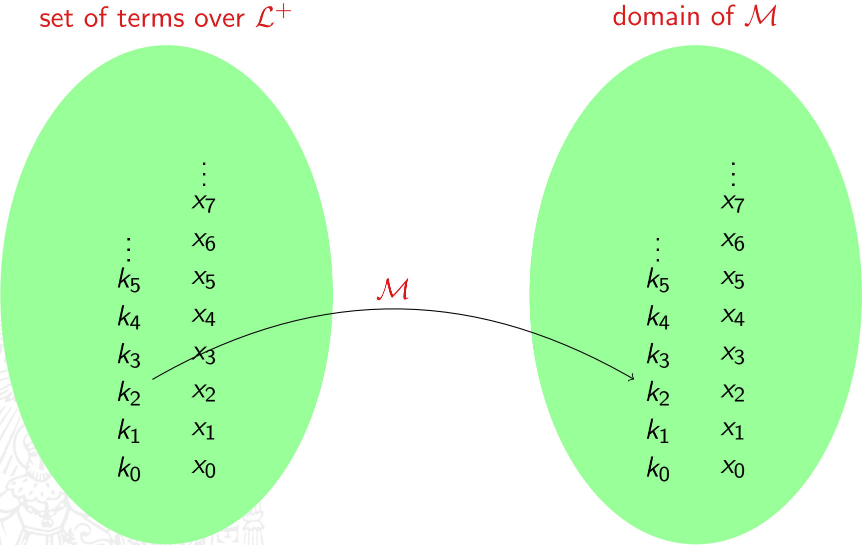

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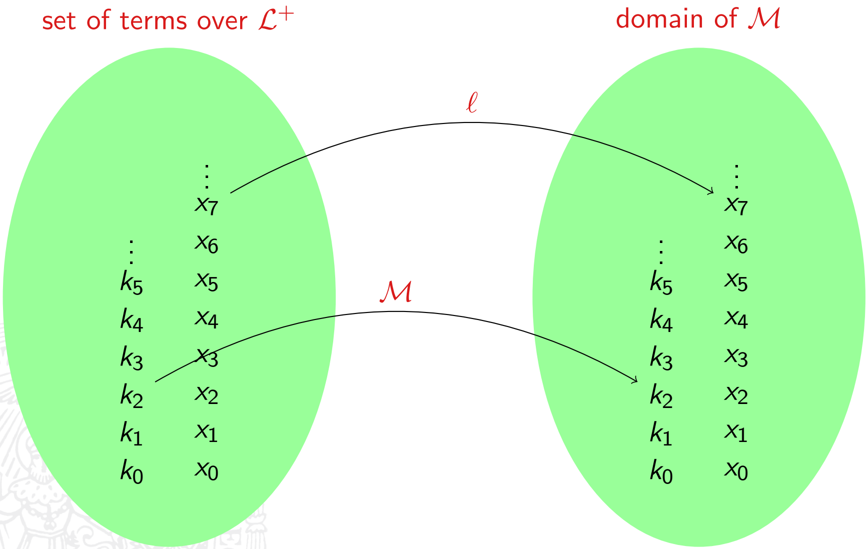
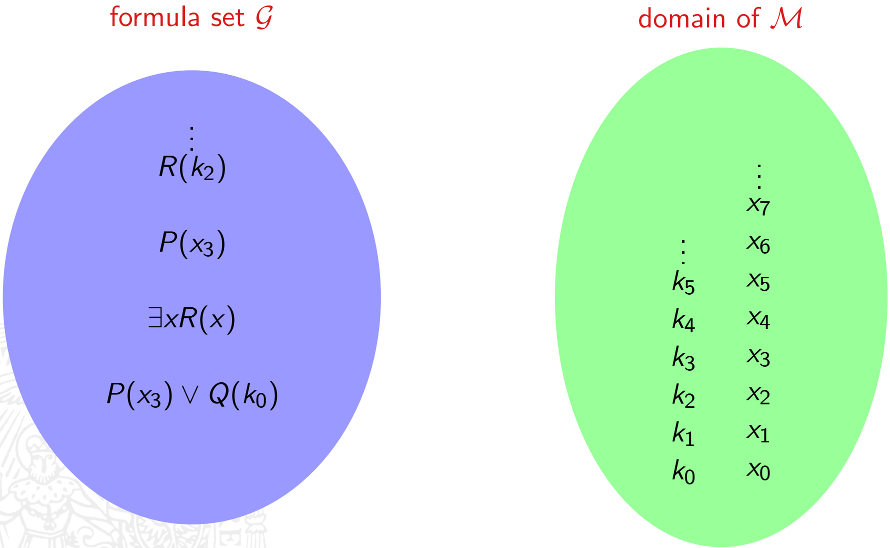


Model Construction in a Picture



Model Construction in a Picture

formula set $\mathcal{G}$

$\vdots$

$R(k_2)$

$P(x_3)$

$\exists x R(x)$

$P(x_3) \vee Q(k_0)$

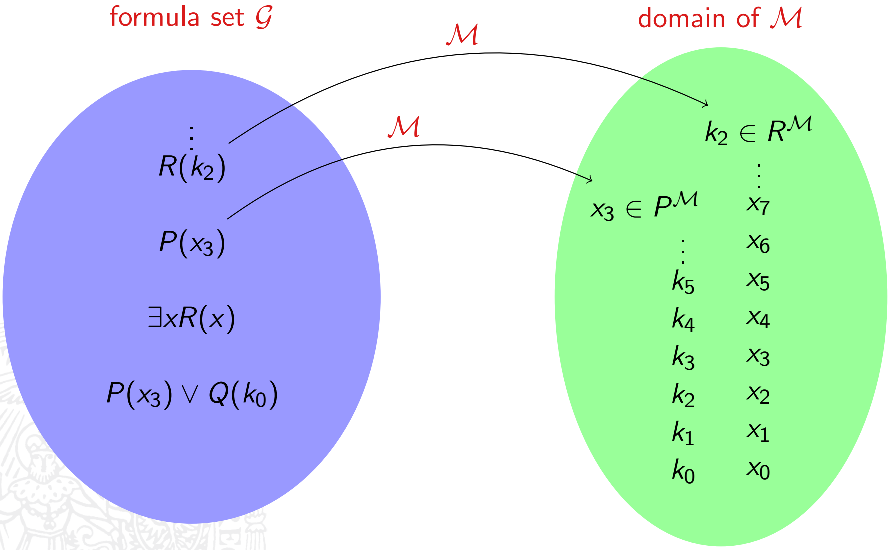
domain of $\mathcal{M}$

$k_2 \in R^{\mathcal{M}}$

$x_3 \in P^{\mathcal{M}}$

$$
\begin{array}{cc}
 & \vdots \\
 & x_7 \\
\vdots & x_6 \\
k_5 & x_5 \\
k_4 & x_4 \\
k_3 & x_3 \\
k_2 & x_2 \\
k_1 & x_1 \\
k_0 & x_0
\end{array}
$$

Model Construction in a Picture



Proof of Lemma ④

(no identity, no function symbols)

- let $\mathcal{L}$ be a language; $\mathcal{L}^+$ extension of $\mathcal{L}$ with infinitely many individual constants
- let S^* be a set of formula sets (of $\mathcal{L}^+$), let S^* admit the satisfaction properties
- $\forall$ formula set $\mathcal{G} \in S^*$ (of $\mathcal{L}$), $\exists\ \mathcal{G}^* \supseteq \mathcal{G}$ (of $\mathcal{L}^+$), such that $\mathcal{G}^*$ fulfils the closure properties

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Proof

- construct sequence of sets belonging to S^*

$$\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots \qquad \mathcal{G}_n \subseteq \mathcal{G}_{n+1}$$

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- $\mathcal{G}_n$ is constructed in step n
- set $\mathcal{G}^* = \bigcup_{n \geqslant 0} \mathcal{G}_n$
- closure properties induce (infinitely many) demands

Proof (cont'd)

Demands

1. no formula F and $\neg F$ in $\mathcal{G}_n$ for all $n \geqslant 0$

Proof (cont'd)

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5. if $\exists x F(x) \in \mathcal{G}_n$, then $\exists$ term t, $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F(t)\}$

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7. $\forall$ terms t, $\exists k \geqslant n$ such that $t = t \in \mathcal{G}_k$

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8. if $F(s) \in \mathcal{G}_n$, and $s = t \in \mathcal{G}_n$, $\exists k \geqslant n$ $F(t) \in \mathcal{G}_k$

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Claim

all demands can be granted, in particular the satisfaction properties guarantee that any demand can be met

Proof (cont'd)

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- consider Demand 5:
  if $\exists x F(x) \in \mathcal{G}_n$, then $\exists$ term t, $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F(t)\}$

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- we use that S^* fulfils the satisfaction properties (c is fresh):

$$\exists x F(x) \in \mathcal{G}_n \in S^* \Rightarrow \mathcal{G}_n \cup \{F(c)\} \in S^*$$

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- we fulfil demand by setting (at step k)

$$\mathcal{G}_{k+1} := \mathcal{G}_k \cup \{F(c)\} \qquad \text{for fresh } c$$

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Claim: $\exists$ fair strategy

- assign a pair (i, n) to each demand except Demand 6
  assign triple $(i, n, \ulcorner t \urcorner)$ to Demand 6, i is the number of the demand raised at step n, $\ulcorner t \urcorner$ Gödel number of t

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Saturation of $\mathcal{G}$ in a Picture

formula set $\mathcal{G} = \mathcal{G}_0$

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$$\exists x R(x)$$

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Generalisation I: Function Constants

Lemma ③ (revisited)

1. let $\mathcal{G}$ be a formula set admitting the closure properties
2. suppose that $\mathcal{L}$ is free of the equality symbol
3. then $\exists$ interpretation $\mathcal{M}$ in which every element of the domain is the denotation of some term
4. $\mathcal{M} \models \mathcal{G}$

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1. $t_1, \dots, t_n$ elements of $\mathcal{M}$ and f an n-ary function symbol in $\mathcal{L}$
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this extends model existence to first-order logic (without $=$)

Generalisation II: Equality

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Proof.

1. suppose $(s = t) \in \mathcal{G}$, where s and t are syntactically different

Generalisation II: Equality

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1. let $\mathcal{G}$ be a formula set admitting the closure properties
2. then $\exists$ interpretation $\mathcal{M}$ in which every element of the domain is the denotation of some term
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2. for $\mathcal{M}$ according to the original construction, we have $\mathcal{M} \not\models s = t$

Generalisation II: Equality

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4. consider the set $\mathcal{E}$ of all equations induced by $\mathcal{G}$:

$$\mathcal{E} = \{s = t \mid \mathcal{G} \models s = t\}$$

Proof (cont'd).

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Proof (cont'd).

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this extends model existence to full first-order logic

Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, Löwenheim-Skolem, compactness, model existence theorem, completeness, natural deduction, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

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Natural Deduction for First-Order Logic

	introduction	elimination
$\wedge$	$\dfrac{E \quad F}{E \wedge F}\ \wedge\!:\mathrm{i}$	$\dfrac{E \wedge F}{E}\ \wedge\!:\mathrm{e} \qquad \dfrac{E \wedge F}{F}\ \wedge\!:\mathrm{e}$
$\vee$	$\dfrac{E}{E \vee F}\ \vee\!:\mathrm{i} \qquad \dfrac{F}{E \vee F}\ \vee\!:\mathrm{i}$	$\dfrac{E \vee F \quad \boxed{\begin{array}{c} E \\ \vdots \\ G \end{array}} \quad \boxed{\begin{array}{c} F \\ \vdots \\ G \end{array}}}{G}\ \vee\!:\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c} E \\ \vdots \\ F \end{array}}}{E \rightarrow F}\ \rightarrow\!:\mathrm{i}$	$\dfrac{E \quad E \rightarrow F}{F}\ \rightarrow\!:\mathrm{e}$

Natural Deduction Extended

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{array}{c} E \\ \vdots \\ \bot \end{array}}}{\neg E}\ \neg\colon \mathrm{i}$	$\dfrac{F \quad \neg F}{\bot}\ \neg\colon \mathrm{e}$
$\bot$		$\dfrac{\bot}{F}\ \bot\colon \mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg F}{F}\ \neg\neg\colon \mathrm{e}$
$=$	$\dfrac{}{t = t}\ =\colon \mathrm{i}$	$\dfrac{s = t \quad F(s)}{F(t)}\ =\colon \mathrm{e}$

Natural Deduction Quantifier Rules

	introduction	elimination
$\exists$	$\dfrac{F(t)}{\exists x F(x)}\ \exists\colon \mathrm{i}$	$\dfrac{\exists x F(x) \quad \boxed{\begin{array}{ll} x & F(x) \\ & \vdots \\ & G \end{array}}}{G}\ \exists\colon \mathrm{e}$
$\forall$	$\dfrac{\boxed{\begin{array}{ll} x & \\ & \vdots \\ & F(x) \end{array}}}{\forall x F(x)}\ \forall\colon \mathrm{i}$	$\dfrac{\forall x F(x)}{F(t)}\ \forall\colon \mathrm{e}$

variable x in $\exists$: e, $\forall$: i local to box

Example

1		$\exists x P(x)$	premise
2		$\forall x \forall y (P(x) \to Q(y))$	premise
3	y		
4	x	$P(x)$	assumption
5		$\forall y (P(x) \to Q(y))$	$2, \forall$: e
6		$P(x) \to Q(y)$	$5, \forall$: e
7		$Q(y)$	$4, 6, \to$: e
8		$Q(y)$	$1, 4-7, \exists$: e
9		$\forall y Q(y)$	$3-8, \forall$: i

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hence we have

$$\exists x P(x), \forall x \forall y (P(x) \to Q(y)) \vdash \forall y Q(y)$$

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hence we have

provability relation

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Gödel's Completeness Theorem

Definition

let $\mathcal{G}$ be a set of formulas, F a formula

- if $\exists$ a natural deduction proof from of F from finite $\mathcal{G}_0 \subseteq \mathcal{G}$, we write $\mathcal{G} \vdash F$

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Theorem

first-order logic is sound and complete: $\mathcal{G} \models F \iff \mathcal{G} \vdash F$

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- the set S of consistent sets of formulas admit the satisfactions properties

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- the set S of consistent sets of formulas admit the satisfactions properties
- by the model existence theorem any $\mathcal{G} \in S$ is satisfiable ∎

Soundness Theorem

first-order logic is sound

$$\mathcal{G} \models F \Leftarrow \mathcal{G} \vdash F$$

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Lemma ⑥

model existence

completeness

Soundness Theorem

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S set of consistent sets $\Rightarrow$ S admits satisfaction properties

Lemma ⑥

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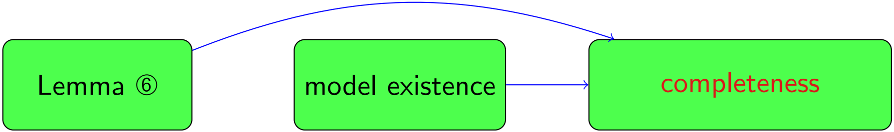

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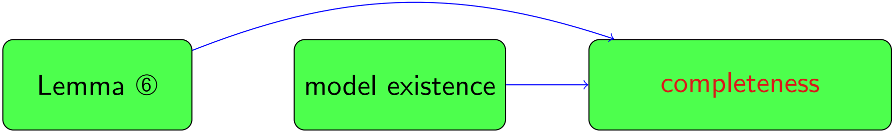

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the set S of all consistent set of formulas has the satisfaction properties

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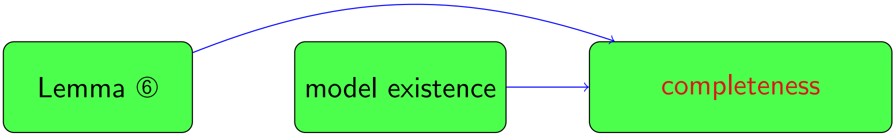

Lemma

$\mathcal{G} \vdash F$ *iff* $\mathcal{G} \cup \{\neg F\}$ *is inconsistent*

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