Automated Reasoning

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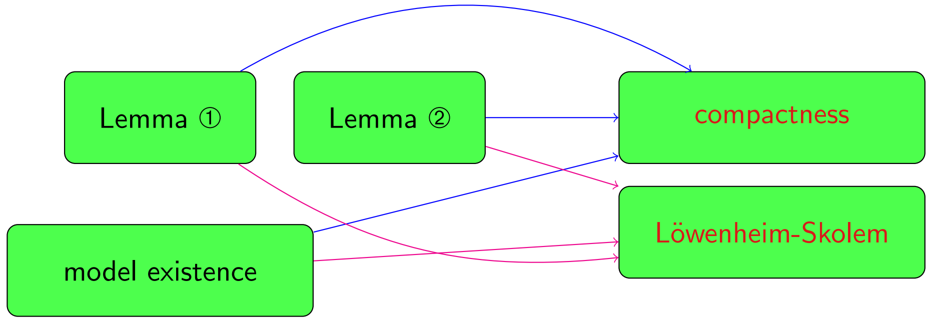
Summary Last Lecture

Compactness Theorem

if every finite subset of a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a model

Löwenheim-Skolem Theorem

if a set of formulas $\mathcal{G}$ has a model, then $\mathcal{G}$ has a countable model



Lemma ①

let S be the set of satisfiable sets of formulas; pick $\mathcal{G} \in S$

1. *if $\mathcal{G}_0 \subseteq \mathcal{G}$, then $\mathcal{G}_0 \in S$*
2. *no formula F and $\neg F$ in $\mathcal{G}$*
3. *if $\neg\neg F \in \mathcal{G}$, then $\mathcal{G} \cup \{F\} \in S$*
4. *if $(E \vee F) \in \mathcal{G}$, then $\mathcal{G} \cup \{E\} \in S$ or $\mathcal{G} \cup \{F\} \in S$*
5. *if $\neg(E \vee F) \in \mathcal{G}$, then $\mathcal{G} \cup \{\neg E\} \in S$ and $\mathcal{G} \cup \{\neg F\} \in S$*
6. *if $\exists x F(x) \in \mathcal{G}$, the constant c doesn't occur in $\mathcal{G}$, then $\mathcal{G} \cup \{F(c)\} \in S$*
7. *if $\neg\exists x F(x) \in \mathcal{G}$, then $\forall$ terms t, $\mathcal{G} \cup \{\neg F(t)\} \in S$*
8. *for any term t, $\mathcal{G} \cup \{t = t\} \in S$*
9. *if $\{F(s), s = t\} \subseteq \mathcal{G}$, then $\mathcal{G} \cup \{F(t)\} \in S$*

Definition

we call the properties (of S) in the lemma satisfaction properties

Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, Löwenheim-Skolem, compactness, model existence theorem, completeness, natural deduction, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

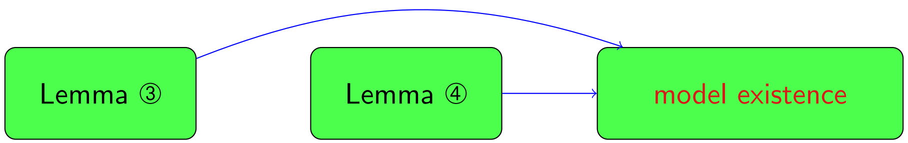
Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

Model Existence

$\mathcal{L}$ base language; $\mathcal{L}^+ \supseteq \mathcal{L}$ infinitely many new individual constants

Theorem (Model Existence Theorem)

1. *if S^* is a set of formula sets of $\mathcal{L}^+$ having the satisfaction properties, then $\forall$ formula sets $\mathcal{G} \in S^*$ of $\mathcal{L}$, $\exists$ $\mathcal{M}$, $\mathcal{M} \models \mathcal{G}$*
2. *$\forall$ elements m of $\mathcal{M}$: m denotes term in $\mathcal{L}^+$*



Closure Properties

Lemma

the set $\mathcal{G}$ of formulas that are true in $\mathcal{M}$ admit

1. *no formula F and $\neg F$ in $\mathcal{G}$*
2. *if $\neg\neg F \in \mathcal{G}$, then $F \in \mathcal{G}$*
3. *if $(E \vee F) \in \mathcal{G}$, then $E \in \mathcal{G}$ or $F \in \mathcal{G}$*
4. *if $\neg(E \vee F) \in \mathcal{G}$, then $\neg E \in \mathcal{G}$ and $\neg F \in \mathcal{G}$*
5. *if $\exists x F(x) \in \mathcal{G}$, then $\exists$ term t (of $\mathcal{L}^+$), $F(t) \in \mathcal{G}$*
6. *if $\neg\exists x F(x) \in \mathcal{G}$, then $\forall$ term t (of $\mathcal{L}^+$), $\neg F(t) \in \mathcal{G}$*
7. *$\forall$ term t (of $\mathcal{L}^+$), $t = t \in \mathcal{G}$*
8. *if $F(s) \in \mathcal{G}$, $s = t \in \mathcal{G}$, then $F(t) \in \mathcal{G}$*

Definition

we call the properties of $\mathcal{G}$ closure properties

Lemma ③

1. let $\mathcal{G}$ be a formula set admitting the closure properties
2. then $\exists$ interpretation $\mathcal{M}$ in which every element of the domain is the denotation of some term
3. $\mathcal{M} \models \mathcal{G}$

Lemma ④

1. let $\mathcal{L}$ be a language; $\mathcal{L}^+$ extension of $\mathcal{L}$ with infinitely many individual constants
2. let S^* be a set of formula sets (of $\mathcal{L}^+$), let S^* admit the satisfaction properties
3. $\forall$ formula set $\mathcal{G} \in S^*$ (of $\mathcal{L}$), $\exists$ $\mathcal{G}^* \supseteq \mathcal{G}$ (of $\mathcal{L}^+$), such that $\mathcal{G}^*$ fulfils the closure properties

Proof of Model Existence

by Lemma ④ and Lemma ③ ■

Proof of Lemma ③

(no identity, no function symbols)

- let $\mathcal{G}$ be a formula set admitting the closure properties
- then $\exists$ interpretation $\mathcal{M}$ in which every element of the domain is the denotation of some term
- $\mathcal{M} \models \mathcal{G}$

Proof

1. the domain of $\mathcal{M}$ is the set of terms (of $\mathcal{L}^+$)
2. $\forall$ constants c
$$c^{\mathcal{M}} := c$$
3. $\forall$ predicate constant P, $\forall$ terms $t_1, \dots, t_n$:
$$(t_1, \dots, t_n) \in P^{\mathcal{M}} \iff P(t_1, \dots, t_n) \in \mathcal{G}$$
4. $\forall$ variables x: $\ell(x) := x$

Proof (cont'd)

5 definition of $\mathcal{M}$ takes care of the demand that every element of its domain is the denotation of a term

6 we claim $\forall$ formulas F: $F \in \mathcal{G} \Rightarrow \mathcal{M} \models F$

Claim: $F \in \mathcal{G} \Rightarrow \mathcal{M} \models F$

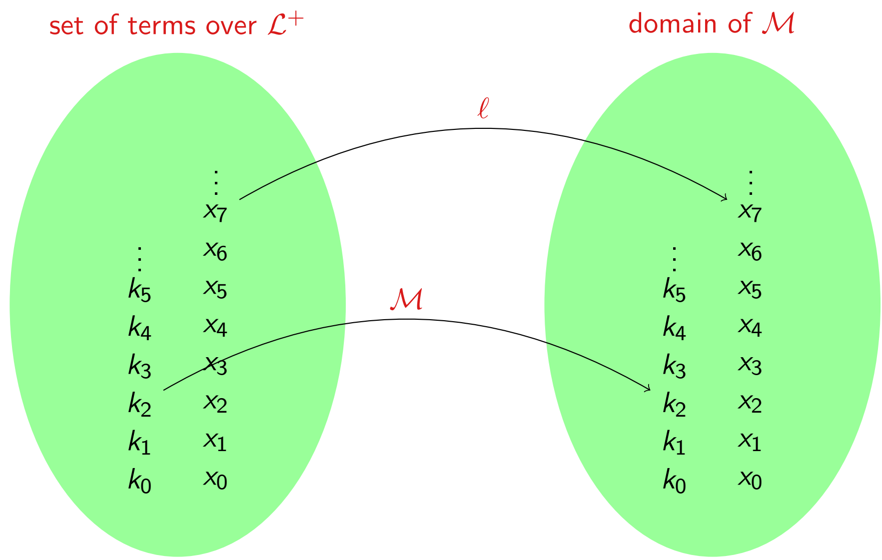
we show the claim by induction on F:

- for the base case, let $F = P(t_1, \dots, t_n)$, if $F \in \mathcal{G}$, then by definition $(t_1, \dots, t_n) \in P^{\mathcal{M}}$; hence $\mathcal{M} \models F$
- for the step case, we assume $F = \exists x G(x)$ and $F \in \mathcal{G}$; the other cases are similar

  by assumption $\mathcal{G}$ fulfils the closure properties, hence there exists a term t such that $G(t) \in \mathcal{G}$

  by induction hypothesis: $\mathcal{M} \models G(t)$ and thus $\mathcal{M} \models \exists x G(x)$ ∎

Model Construction in a Picture



Proof of Lemma ④

(no identity, no function symbols)

- let $\mathcal{L}$ be a language; $\mathcal{L}^+$ extension of $\mathcal{L}$ with infinitely many individual constants
- let S^* be a set of formula sets (of $\mathcal{L}^+$), let S^* admit the satisfaction properties
- $\forall$ formula set $\mathcal{G} \in S^*$ (of $\mathcal{L}$), $\exists\, \mathcal{G}^* \supseteq \mathcal{G}$ (of $\mathcal{L}^+$), such that $\mathcal{G}^*$ fulfils the closure properties

Proof

- construct sequence of sets belonging to S^*

$$\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots \qquad \mathcal{G}_n \subseteq \mathcal{G}_{n+1}$$

- $\mathcal{G}_n$ is constructed in step n
- set $\mathcal{G}^* = \bigcup_{n \geqslant 0} \mathcal{G}_n$
- closure properties induce (infinitely many) demands

Proof (cont'd)

Demands

1 no formula F and $\neg F$ in $\mathcal{G}_n$ for all $n \geqslant 0$

2 if $\neg\neg F \in \mathcal{G}_n$, then $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F\}$

3 if $(E \vee F) \in \mathcal{G}_n$, then $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{E\}$ or $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F\}$

4 if $\neg(E \vee F) \in \mathcal{G}_n$, then $\exists k_1, k_2 \geqslant n$, $\mathcal{G}_{k_1+1} = \mathcal{G}_{k_1} \cup \{\neg E\}$ and $\mathcal{G}_{k_2+1} = \mathcal{G}_{k_2} \cup \{\neg F\}$

5 if $\exists x F(x) \in \mathcal{G}_n$, then $\exists$ term t, $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F(t)\}$

6 if $\neg\exists x F(x) \in \mathcal{G}_n$, then $\forall$ term t, $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{\neg F(t)\}$

7 $\forall$ terms t, $\exists k \geqslant n$ such that $t = t \in \mathcal{G}_k$

8 if $F(s) \in \mathcal{G}_n$, and $s = t \in \mathcal{G}_n$, $\exists k \geqslant n$ $F(t) \in \mathcal{G}_k$

Claim

all demands can be granted, in particular the satisfaction properties guarantee that any demand can be met

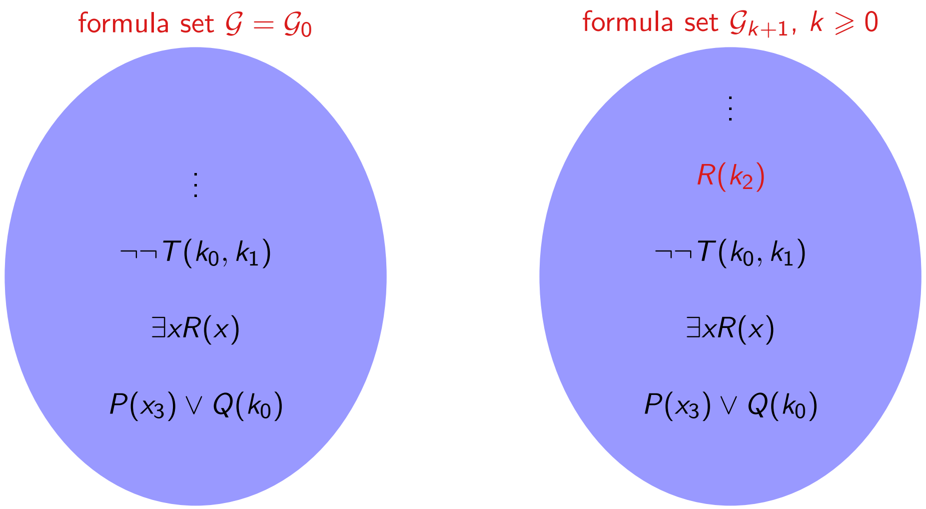
Proof (cont'd)

- consider Demand 5:
  if $\exists x F(x) \in \mathcal{G}_n$, then $\exists$ term t, $\exists k \geqslant n$, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F(t)\}$
- we use that S^* fulfils the satisfaction properties (c is fresh):
$$\exists x F(x) \in \mathcal{G}_n \in S^* \Rightarrow \forall k \geqslant n \; \mathcal{G}_k \cup \{F(c)\} \in S^*$$
- we fulfil demand by setting (at step k)
$$\mathcal{G}_{k+1} := \mathcal{G}_k \cup \{F(c)\} \qquad \text{for fresh } c$$
- similar for the Demands 2–8

Claim: $\exists$ fair strategy

- assign a pair (i, n) to each demand except Demand 6
  assign triple $(i, n, \ulcorner t \urcorner)$ to Demand 6, i is the number of the demand raised at step n, $\ulcorner t \urcorner$ Gödel number of t
- enumerate all pairs or triples and encode them as number k
- in step k we grant the demand raised at step n ∎

Saturation of $\mathcal{G}$ in a Picture



$\exists x F(x) \in \mathcal{G}_n$, then $\exists k \geqslant n$, $\exists$ term t, $\mathcal{G}_{k+1} = \mathcal{G}_k \cup \{F(t)\}$

Generalisation I: Function Constants

Lemma ③ (revisited)

1. let $\mathcal{G}$ be a formula set admitting the closure properties
2. suppose that $\mathcal{L}$ is free of the equality symbol
3. then $\exists$ interpretation $\mathcal{M}$ in which every element of the domain is the denotation of some term
4. $\mathcal{M} \models \mathcal{G}$

Proof.

1. $t_1, \dots, t_n$ elements of $\mathcal{M}$ and f an n-ary function symbol in $\mathcal{L}$
2. define: $f^{\mathcal{M}}(t_1, \dots, t_n) := f(t_1, \dots, t_n)$
3. following the earlier proof, we verify $\mathcal{M} \models \mathcal{G}$

this extends model existence to first-order logic (without $=$)

Generalisation II: Equality

Lemma ③ (revisited again)

1. let $\mathcal{G}$ be a formula set admitting the closure properties
2. then $\exists$ interpretation $\mathcal{M}$ in which every element of the domain is the denotation of some term
3. $\mathcal{M} \models \mathcal{G}$

Proof.

1. suppose $(s = t) \in \mathcal{G}$, where s and t are syntactically different
2. for $\mathcal{M}$ according to the original construction, we have $\mathcal{M} \not\models s = t$
3. define a variant of the model $\mathcal{M}$, denoted as $\mathcal{M}'$
4. consider the set $\mathcal{E}$ of all equations induced by $\mathcal{G}$:
$$\mathcal{E} = \{s = t \mid \mathcal{G} \models s = t\}$$

Proof (cont'd).

5. $\mathcal{E}$ gives rise to an equivalence relation $\sim$
6. domain of $\mathcal{M}'$ is set of equivalent classes of terms of $\mathcal{L}^+$
7. $[t]_\sim$ denotes the equivalence class of t
8. definition of the structure underlying $\mathcal{M}'$:

$$\begin{array}{ll} f^{\mathcal{M}}([t_1]_\sim, \dots, [t_n]_\sim) = [f(t_1, \dots, t_n)]_\sim & f \text{ is } n\text{-ary function} \\ P^{\mathcal{M}}([t_1]_\sim, \dots, [t_n]_\sim) \iff P(t_1, \dots, t_n) \in \mathcal{G} & P \text{ is } n\text{-ary predicate} \end{array}$$

9. from this $\mathcal{M}' \models \mathcal{G}$ ∎

this extends model existence to full first-order logic

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Natural Deduction for First-Order Logic

	introduction	elimination
$\wedge$	$\dfrac{E \quad F}{E \wedge F}\ \wedge\colon \mathrm{i}$	$\dfrac{E \wedge F}{E}\ \wedge\colon \mathrm{e} \qquad \dfrac{E \wedge F}{F}\ \wedge\colon \mathrm{e}$
$\vee$	$\dfrac{E}{E \vee F}\ \vee\colon \mathrm{i} \qquad \dfrac{F}{E \vee F}\ \vee\colon \mathrm{i}$	$\dfrac{E \vee F \quad \begin{array}{\|c\|}\hline E \\ \vdots \\ G \\ \hline\end{array} \quad \begin{array}{\|c\|}\hline F \\ \vdots \\ G \\ \hline\end{array}}{G}\ \vee\colon \mathrm{e}$
$\rightarrow$	$\dfrac{\begin{array}{\|c\|}\hline E \\ \vdots \\ F \\ \hline\end{array}}{E \rightarrow F}\ \rightarrow\colon \mathrm{i}$	$\dfrac{E \quad E \rightarrow F}{F}\ \rightarrow\colon \mathrm{e}$

Natural Deduction Extended

	introduction	elimination
$\neg$	$\dfrac{\begin{array}{\|c\|}\hline E \\ \vdots \\ \bot \\ \hline\end{array}}{\neg E}\ \neg\colon \mathrm{i}$	$\dfrac{F \quad \neg F}{\bot}\ \neg\colon \mathrm{e}$
$\bot$		$\dfrac{\bot}{F}\ \bot\colon \mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg F}{F}\ \neg\neg\colon \mathrm{e}$
$=$	$\dfrac{}{t = t}\ =\colon \mathrm{i}$	$\dfrac{s = t \quad F(s)}{F(t)}\ =\colon \mathrm{e}$

Natural Deduction Quantifier Rules

	introduction	elimination
$\exists$	$\dfrac{F(t)}{\exists x F(x)}\ \exists\colon i$	$\dfrac{\exists x F(x) \quad \boxed{\begin{array}{ll} x & F(x) \\ & \vdots \\ & G \end{array}}}{G}\ \exists\colon e$
$\forall$	$\dfrac{\boxed{\begin{array}{ll} x & \\ & \vdots \\ & F(x) \end{array}}}{\forall x F(x)}\ \forall\colon i$	$\dfrac{\forall x F(x)}{F(t)}\ \forall\colon e$

variable x in $\exists\colon$ e, $\forall\colon$ i local to box

Example

1		$\exists x \mathsf{P}(x)$	premise
2		$\forall x \forall y (\mathsf{P}(x) \to \mathsf{Q}(y))$	premise
3	y		
4	x	$\mathsf{P}(x)$	assumption
5		$\forall y (\mathsf{P}(x) \to \mathsf{Q}(y))$	$2, \forall\colon$ e
6		$\mathsf{P}(x) \to \mathsf{Q}(y)$	$5, \forall\colon$ e
7		$\mathsf{Q}(y)$	$4, 6, \to\colon$ e
8		$\mathsf{Q}(y)$	$1, 4-7, \exists\colon$ e
9		$\forall y \mathsf{Q}(y)$	$3-8, \forall\colon$ i

hence we have

provability relation

$$\exists x \mathsf{P}(x), \forall x \forall y (\mathsf{P}(x) \to \mathsf{Q}(y)) \vdash \forall y \mathsf{Q}(y)$$

Gödel's Completeness Theorem

Definition

let $\mathcal{G}$ be a set of formulas, F a formula

- if $\exists$ a natural deduction proof from of F from finite $\mathcal{G}_0 \subseteq \mathcal{G}$, we write $\mathcal{G} \vdash F$
- if $\neg\,\exists$ proof of $\bot$ from $\mathcal{G}$, we say $\mathcal{G}$ is consistent, otherwise inconsistent

Theorem

first-order logic is sound and complete: $\mathcal{G} \models F \iff \mathcal{G} \vdash F$

Proof Idea

- the set S of consistent sets of formulas admit the satisfactions properties
- by the model existence theorem any $\mathcal{G} \in S$ is satisfiable ∎

Soundness Theorem

first-order logic is sound

$$\mathcal{G} \models F \Leftarrow \mathcal{G} \vdash F$$

Lemma ⑥ → completeness

model existence → completeness

Lemma

$\mathcal{G} \vdash F$ *iff* $\mathcal{G} \cup \{\neg F\}$ *is inconsistent*

Lemma ⑥

the set S of all consistent set of formulas has the satisfaction properties

Completeness Theorem

first-order logic is complete

$$\mathcal{G} \models F \Rightarrow \mathcal{G} \vdash F$$

Proof.

1. wlog $\exists$ finite $\mathcal{G}_0 \subseteq \mathcal{G}$, $\mathcal{G}_0 \models F$, recall $\mathcal{G} \models F$ iff $\neg\,\mathsf{Sat}(\mathcal{G} \cup \{\neg F\})$ if ($\exists$ finite $\mathcal{G}_0 \subseteq \mathcal{G}$ $\neg\,\mathsf{Sat}(\mathcal{G}_0 \cup \{\neg F\})$) iff $\mathcal{G}_0 \models F$
2. suppose $\mathcal{G}_0 \nvdash F$, we have to show $\mathcal{G}_0 \not\models F$
3. suppose $\mathcal{G}_0 \cup \{\neg F\}$ is consistent, then $\mathcal{G}_0 \cup \{\neg F\}$ is satisfiable
4. Lemma ⑥ yields that the set S of consistent formulas sets fulfils the satisfaction properties
5. model existence yields that $\forall\, \mathcal{H} \in S$, $\mathcal{H}$ satisfiable
6. as $\mathcal{G}_0 \cup \{\neg F\} \in S$, $\mathcal{G}_0 \cup \{\neg F\}$ satisfiable ∎