Automated Reasoning

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Summary Last Lecture

Theorem (Model Existence Theorem)

1. *if S^* is a set of formula sets of $\mathcal{L}^+$ having the satisfaction properties, then $\forall$ formula sets $\mathcal{G} \in S^*$ of $\mathcal{L}$, $\exists$ $\mathcal{M}$, $\mathcal{M} \models \mathcal{G}$*
2. *$\forall$ elements m of $\mathcal{M}$: m denotes term in $\mathcal{L}^+$*

Definition

let $\mathcal{G}$ be a set of formulas, F a formula

- if $\exists$ a natural deduction proof from of F from finite $\mathcal{G}_0 \subseteq \mathcal{G}$, we write $\mathcal{G} \vdash F$

Theorem

first-order logic is sound and complete: $\mathcal{G} \models F \iff \mathcal{G} \vdash F$

Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, undecidability of first-order, Löwenheim-Skolem, compactness, model existence theorem, natural deduction, completeness, sequent calculus, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

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Definition

- the expression $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ is called a sequent
- intuitively this means $A_1 \wedge \cdots \wedge A_n \rightarrow B_1 \vee \cdots \vee B_m$

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Example

the following expression is a sequent

$$\exists x \mathrm{P}(x), \forall x \forall y (\mathrm{P}(x) \rightarrow \mathrm{Q}(y)) \Rightarrow \forall y \mathrm{Q}(y)$$

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Definitions

- the formulas A_i, B_j are called sequent formulas; let $\Gamma = \{A_1, \ldots, A_n\}$, $\Delta = \{B_1, \ldots, B_m\}$, then Γ is the antecedent, Δ the succedent
- sequences of sequent formulas are considered as multisets
- Greek capital letters $\Gamma, \Delta, \Lambda, \ldots$ are used to denote multisets of sequent formulas

Rules of Sequent Calculus

	left	right
$\wedge$	$\dfrac{E, \Gamma \Rightarrow \Delta}{E \wedge F, \Gamma \Rightarrow \Delta}\ \wedge : \mathsf{l}$ $\dfrac{F, \Gamma \Rightarrow \Delta}{E \wedge F, \Gamma \Rightarrow \Delta}\ \wedge : \mathsf{l}$	$\dfrac{\Gamma \Rightarrow \Delta, E \quad \Gamma \Rightarrow \Delta, F}{\Gamma \Rightarrow \Delta, E \wedge F}\ \wedge : \mathsf{r}$
$\vee$	$\dfrac{E, \Gamma \Rightarrow \Delta \quad F, \Gamma \Rightarrow \Delta}{E \vee F, \Gamma \Rightarrow \Delta}\ \vee : \mathsf{l}$	$\dfrac{\Gamma \Rightarrow \Delta, E}{\Gamma \Rightarrow \Delta, E \vee F}\ \vee : \mathsf{r}$ $\dfrac{\Gamma \Rightarrow \Delta, F}{\Gamma \Rightarrow \Delta, E \vee F}\ \vee : \mathsf{r}$
$\rightarrow$	$\dfrac{\Gamma \Rightarrow \Delta, E \quad F, \Gamma \Rightarrow \Delta}{E \rightarrow F, \Gamma \Rightarrow \Delta}\ \rightarrow : \mathsf{l}$	$\dfrac{\Gamma, E \Rightarrow \Delta, F}{\Gamma \Rightarrow \Delta, E \rightarrow F}\ \rightarrow : \mathsf{l}$

Sequent Calculus (cont'd)

	left	right
$\neg$	$\dfrac{\Gamma \Rightarrow \Delta, E}{\neg E, \Gamma \Rightarrow \Delta}\ \neg\colon \mathsf{l}$	$\dfrac{E, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg E}\ \neg\colon \mathsf{r}$
$=$	$\Rightarrow t = t$	$s_1 = t_1, \ldots, s_n = t_n \Rightarrow f(\overline{s}) = f(\overline{t})$ $s_1 = t_1, \ldots, s_n = t_n \Rightarrow P(\overline{s}) = P(\overline{t})$
$\exists$	$\dfrac{F(x), \Gamma \Rightarrow \Delta}{\exists x F(x), \Gamma \Rightarrow \Delta}\ \exists\colon \mathsf{l}$	$\dfrac{\Gamma \Rightarrow \Delta, F(t)}{\Gamma \Rightarrow \Delta, \exists x F(x)}\ \exists\colon \mathsf{r}$
$\forall$	$\dfrac{F(t), \Gamma \Rightarrow \Delta}{\forall x F(x), \Gamma \Rightarrow \Delta}\ \forall\colon \mathsf{l}$	$\dfrac{\Gamma \Rightarrow \Delta, F(x)}{\Gamma \Rightarrow \Delta, \forall x F(x)}\ \forall\colon \mathsf{r}$

variable x in $\exists\colon \mathsf{l}$, $\forall\colon \mathsf{r}$ must not occur free in lower sequent
(eigenvariable condition)

Sequent Calculus Structural Rules

	left	right
axiom and cut	$A \Rightarrow A$	$\dfrac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$
contraction	$\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$ c: l	$\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$ c: r
weakening	$\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$ w: l	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$ w: r

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Observation

we note the link between elimination (introduction) rules in natural deduction and left (right) rules in sequent calculus

Example revisited

Example

1		$\exists x \mathrm{P}(x)$	premise
2		$\forall x \forall y (\mathrm{P}(x) \rightarrow \mathrm{Q}(y))$	premise
3	y		
4	x	$\mathrm{P}(x)$	assumption
5		$\forall y (\mathrm{P}(x) \rightarrow \mathrm{Q}(y))$	$2, \forall$: e
6		$\mathrm{P}(x) \rightarrow \mathrm{Q}(y)$	$5, \forall$: e
7		$\mathrm{Q}(y)$	$4, 6, \rightarrow$: e
8		$\mathrm{Q}(y)$	$1, 4-7, \exists$: e
9		$\forall y \mathrm{Q}(y)$	$3-8, \forall$: i

Example revisited

Example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathsf{P}(x) \Rightarrow \mathsf{P}(x)}{\mathsf{P}(x) \Rightarrow \mathsf{Q}(y), \mathsf{P}(x)}\ \mathsf{w}\colon \mathsf{l} \qquad \cfrac{\mathsf{Q}(y) \Rightarrow \mathsf{Q}(y)}{\mathsf{P}(x), \mathsf{Q}(y) \Rightarrow \mathsf{Q}(y)}\ \mathsf{w}\colon \mathsf{l}}{\mathsf{P}(x), \mathsf{P}(x) \rightarrow \mathsf{Q}(y) \Rightarrow \mathsf{Q}(y)}\ \rightarrow\colon \mathsf{l}}{\mathsf{P}(x), \forall y(\mathsf{P}(x) \rightarrow \mathsf{Q}(y)) \Rightarrow \mathsf{Q}(y)}\ \forall\colon \mathsf{l}}{\mathsf{P}(x), \forall x \forall y(\mathsf{P}(x) \rightarrow \mathsf{Q}(y)) \Rightarrow \mathsf{Q}(y)}\ \forall\colon \mathsf{l}}{\exists x \mathsf{P}(x), \forall x \forall y(\mathsf{P}(x) \rightarrow \mathsf{Q}(y)) \Rightarrow \mathsf{Q}(y)}\ \exists\colon \mathsf{l}}{\exists x \mathsf{P}(x), \forall x \forall y(\mathsf{P}(x) \rightarrow \mathsf{Q}(y)) \Rightarrow \forall y \mathsf{Q}(y)}\ \forall\colon \mathsf{r}}{}
$$

Normalisation

Motivation

- consider the following two abstract derivations:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ E\end{matrix} \quad \begin{matrix}\Pi_2 \\ F\end{matrix}}{E \wedge F}\wedge\colon \mathrm{i}}{E}\wedge\colon \mathrm{e} \qquad\qquad\qquad \begin{matrix}\Pi_2 \\ E\end{matrix}$$

- clearly the right derivation can replace the left one
- the situation is called detour
- the rewrite step is called normalisation

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Definition

- process of eliminating all detours is called normalisation
- strong normalisation means that normalisation terminates for all possible reduction sequences

Definition (Minimal Propositional Logic)

- minimal logic contains $\bot$ as truth constant, and $\wedge$, $\vee$, $\rightarrow$
- negation is defined:
$$\neg A := A \rightarrow \bot$$
- natural deduction for minimal logic consists of:
$$\wedge : \mathsf{i}, \wedge : \mathsf{e} \quad \vee : \mathsf{i}, \vee : \mathsf{e} \quad \rightarrow : \mathsf{i}, \rightarrow : \mathsf{e}$$

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Lemma

- *in minimal logic* $\neg A, A \nvdash B$*;*

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Lemma

- *in minimal logic $\neg A, A \nvdash B$; minimal logic is restriction of classical logic (and also of intuitionistic logic)*

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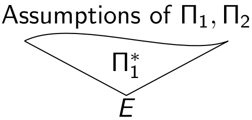
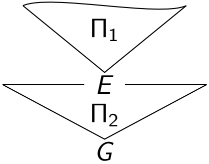
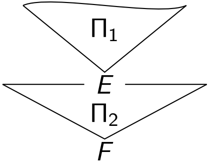
Lemma

- *in minimal logic $\neg A, A \nvdash B$; minimal logic is restriction of classical logic (and also of intuitionistic logic)*
- *to obtain classical logic, we may add the following proof by contradiction (PBC)*

$$\dfrac{\boxed{\begin{array}{l} \neg E \\ \vdots \\ \bot \end{array}}}{E}\ \bot$$

Immediate Reductions

	detour	*contraction*
$\wedge$	$\dfrac{\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ E & F\end{array}}{E \wedge F}\ \wedge:\mathrm{i}}{E}\ \wedge:\mathrm{e}$	
$\vee$	$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ E\end{array}}{E \vee F}\ \vee:\mathrm{i} \quad \begin{array}{c}\Pi_2\\ \boxed{\begin{array}{c}E\\ \vdots\\ G\end{array}}\end{array} \quad \begin{array}{c}\Pi_3\\ \boxed{\begin{array}{c}F\\ \vdots\\ G\end{array}}\end{array}}{G}\ \vee:\mathrm{e}$	
$\to$	$\dfrac{\begin{array}{c}\Pi_1\\ E\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ \boxed{\begin{array}{c}E\\ \vdots\\ F\end{array}}\end{array}}{E \to F}\ \to:\mathrm{i}}{F}\ \to:\mathrm{e}$	

(Strong) Normalisation Theorem

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- Π is immediately reduced to Ψ, if Ψ is obtained by an immediate reduction

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- a reduction sequence is a sequence of proofs $\Pi_1, \dots, \Pi_n$, such that Π_{i+1} is an immediate reduct of Π_i and Π_n is normal

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Theorem (Normalisation and Strong Normalisation)

let Π be a proof in minimal logic

1. $\exists$ *a reduction sequence* $\Pi = \Pi_1, \dots, \Pi_n$
2. $\exists$ *computable upper bound n on the maximal length of any reduction sequence*

Normalisation in General

Theorem (Gentzen, Prawitz)

let Π be a proof in intuitionistic logic; then Π reduces to a normal proof Ψ and any reduction sequence terminates

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Facts

- normalisation or strong normalisation theorem holds for many many logics
- normalisation in natural deduction corresponds to cut-elimination in sequent calculus

Consistency Proofs

Lemma (Subformula Property)

let Π be a normal proof of A, any formula B in Π fulfils one of the following assertions:

1. *B is a subformula of A*
2. *B is (closed) assumption of PBC; $B = \neg C$ and C is a subformula of A*
3. *$B = \perp$ and is used as result of PBC*

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Corollary

$\neg\ \exists$ *normal derivation of* $\perp$

Craig's Interpolation Theorem

Lemma

if sentence $A \to C$ holds, $\exists$ sentence B such that

1. $A \to B$ *and* $B \to C$
2. *all axioms in B occur in both A and C*

Craig's Interpolation Theorem

Lemma

if sentence $A \to C$ holds, $\exists$ sentence B such that

1. $A \to B$ *and* $B \to C$
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if sentence $A \rightarrow C$ holds, $\exists$ sentence B such that

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Example

consider $\underbrace{\exists x F(x) \wedge \exists x \neg F(x)}_{A} \rightarrow \underbrace{\exists x \exists y \; x \neq y}_{C}$ but $\neg \exists$ interpolant B

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Theorem

if sentence $A \rightarrow C$ holds, $\exists$ sentence B such that

1. $A \rightarrow B$ *and* $B \rightarrow C$
2. *all nonlogical constants in B occur in both A and C*

Proof of Craig's Interpolation Theorem

Degnerated Cases

- suppose A is unsatisfiable:

  use $\exists x\ x \neq x$ as interpolant

- suppose C is valid:

  use $\exists x\ x = x$ as interpolant

Proof of Craig's Interpolation Theorem

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Definition

a pair of set of sentences $(\mathcal{G}_1, \mathcal{G}_2)$ is barred by B if

1. $\mathcal{G}_1$ are satisfiable A-sentences, $\mathcal{G}_2$ are satisfiable C-sentences
2. B is both an A-sentence and a C-sentence
3. $\mathcal{G}_1 \models B$ and $\mathcal{G}_2 \models \neg B$

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Definition

a sets of sentences $\mathcal{G}$ admits unbarred division, if

1. $\exists$ pair $(\mathcal{G}_1, \mathcal{G}_2)$ of A-sentences and C-sentences
2. $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are satisfiable
3. no sentence bars $\mathcal{G}_1, \mathcal{G}_2$

Proof of Craig's Interpolation Theorem (no $=$, no functions).

1. assume $\neg\,\exists$ interpolant B

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4. verify that S admits the satisfaction properties, wlog we only show

   let $\mathcal{G} \in S$, if $(E \vee F) \in \mathcal{G}$, then either $\mathcal{G} \cup \{E\} \in S$ or $\mathcal{G} \cup \{F\} \in S$

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6. wlog $(E \vee F) \in \mathcal{G}_1$

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6. wlog $(E \vee F) \in \mathcal{G}_1$
7. it suffices to show that $(\mathcal{G}_1 \cup \{E\}, \mathcal{G}_2)$ or $(\mathcal{G}_1 \cup \{F\}, \mathcal{G}_2)$ forms an unbarred division of $\mathcal{G} \cup \{E\} \in S$

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8. wlog $\mathcal{G}_1 \cup \{E\}$ and $\mathcal{G}_1 \cup \{F\}$ are satisfiable

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7. it suffices to show that $(\mathcal{G}_1 \cup \{E\}, \mathcal{G}_2)$ or $(\mathcal{G}_1 \cup \{F\}, \mathcal{G}_2)$ forms an unbarred division of $\mathcal{G} \cup \{E\} \in S$
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9. assume further both alternatives fail to be unbarred divisions; then we derive a contradiction that $\mathcal{G}$ admits an unbarred division

Proof of Craig's Interpolation Theorem (no $=$, no functions).

1. assume $\neg\,\exists$ interpolant B
2. define collection S of sets of sentences such that $\{A, \neg C\} \in S$ and S will fulfil the satisfaction properties
3. $S =$ collection of sentences $\mathcal{G}$ that admit an unbarred division
4. verify that S admits the satisfaction properties, wlog we only show

   let $\mathcal{G} \in S$, if $(E \vee F) \in \mathcal{G}$, then either $\mathcal{G} \cup \{E\} \in S$ or $\mathcal{G} \cup \{F\} \in S$
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the union $S \cup T$ of two theories S and T is satisfiable iff there is no sentence in S whose negation is in T

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Definitions

- $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ are languages such that $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$
- T_i is theory in $\mathcal{L}_i$ $(i \in \{0, 1, 2\})$

Theorem

if T_1, T_2 are conservative extensions of T_0, then T_3 is a conservative extension of T_0, where $T_3 = \{A \mid T_1 \cup T_2 \models A\}$

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8. this yields that A is theorem of T_0

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6. by the above T_3 is satisfiable, hence $T_1 \cup T_2$ is satisfiable

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Outline of the Lecture

Propositional Logic

short reminder of propositional logic, soundness and completeness theorem, natural deduction, propositional resolution

First Order Logic

introduction, syntax, semantics, undecidability of first-order, Löwenheim-Skolem, compactness, model existence theorem, natural deduction, completeness, sequent calculus, normalisation

Properties of First Order Logic

Craig's Interpolation Theorem, Robinson's Joint Consistency Theorem, Herbrand's Theorem

Limits and Extensions of First Order Logic

Intuitionistic Logic, Curry-Howard Isomorphism, Limits, Second-Order Logic

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Definition (Prenex Normal Form)

1. a formula F is in prenex normal form if it has the form

$$Q_1 x_1 \cdots Q_n x_n \underbrace{G}_{\text{matrix}} \qquad\qquad Q_i \in \{\forall, \exists\}$$

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Example

consider $\forall x F(x) \leftrightarrow G(a)$ or more precisely

$$(\neg\forall x F(x) \vee G(a)) \wedge (\neg G(a) \vee \forall x F(x))$$

one CNF would be

$$\forall x \exists y((\neg F(y) \vee G(a)) \wedge (\neg G(a) \vee F(x)))$$

Theorem

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Theorem

$\forall$ *first-order formula* F, $\exists$ G *such that* G *is in prenex normal form and* $F \equiv G$; *furthermore* G *can be effectively constructed from* F

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$$\neg\forall x F(x) \equiv \exists x \neg F(x)$$
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   $$\forall x_1 \cdots \forall x_{i-1} \exists x_i Q_{i+1} x_{i+1} \cdots Q_m x_m \; G(x_1, \ldots, x_i, \ldots, x_m)$$

   by $s(F'')$

   $$\forall x_1 \cdots \forall x_{i-1} Q_{i+1} x_{i+1} \cdots Q_m x_m \; G(x_1, \ldots, f(x_1, \ldots, x_{i-1}), \ldots, x_m)$$

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Definition

formulas F and G are equivalent for satisfiability ($F \approx G$) whenever F is satisfiable iff G is satisfiable

Theorem

$\forall$ first-order formula, F $\exists$ formula in SNF G such that $F \approx G$; furthermore G can be effectively constructed from F

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- set

$$F = \forall x_1 \cdots \forall x_{i-1} \exists x_i \cdots \mathsf{Q}_m x_m \; G(x_1, \dots, x_m)$$
$$s(F) = \forall x_1 \cdots \forall x_{i-1} \cdots \mathsf{Q}_m x_m \; G(x_1, \dots, f(x_1, \dots, x_{i-1}), \dots, x_m)$$
$$H(x_1, \dots, x_i) = \mathsf{Q}_{i+1} x_{i+1} \cdots \mathsf{Q}_m x_m \; G(x_1, \dots, x_m)$$

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- suppose $F = \forall x_1 \cdots \forall x_{i-1} \exists x_i H(x_1, \dots, x_i)$ is satisfiable
- $\exists$ model $\mathcal{M}$ of F, then $\exists$ expansion $\mathcal{M}^+$ of $\mathcal{M}$ such that

$$\mathcal{M}^+ \models H(x_1, \dots, x_{i-1}, f(x_1, \dots, x_{i-1}))$$

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- $\forall x_1 \cdots \forall x_{i-1} H(x_1, \ldots, f(x_1, \ldots, x_{i-1})) = s(F)$ is satisfiable

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$$H(x_1, \dots, x_i) = \mathsf{Q}_{i+1} x_{i+1} \cdots \mathsf{Q}_m x_m \; G(x_1, \dots, x_m)$$

- suppose $F = \forall x_1 \cdots \forall x_{i-1} \exists x_i H(x_1, \dots, x_i)$ is satisfiable
- $\exists$ model $\mathcal{M}$ of F, then $\exists$ expansion $\mathcal{M}^+$ of $\mathcal{M}$ such that

$$\mathcal{M}^+ \models H(x_1, \dots, x_{i-1}, f(x_1, \dots, x_{i-1}))$$

- $\forall x_1 \cdots \forall x_{i-1} H(x_1, \dots, f(x_1, \dots, x_{i-1})) = s(F)$ is satisfiable

∎

Example

consider $\forall y \forall x (x > y \rightarrow \exists z (x > z \wedge z > y))$; its SNF is

$$\forall y \forall x (\neg(x > y) \vee x > \mathsf{f}(x, y)) \wedge (\neg(x > y) \vee \mathsf{f}(x, y) > y)$$

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let $\mathcal{L} = \{\mathsf{c}, \mathsf{f}, \mathsf{P}\}$, then the Herbrand universe H of $\mathcal{L}$ is

$$H = \{\mathsf{c}, \mathsf{f}(\mathsf{c}), \mathsf{f}(\mathsf{f}(\mathsf{c})), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{c}))), \dots\}$$

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- an interpretation $\mathcal{I}$ (of $\mathcal{L}$) is Herbrand interpretation if
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- note that $\mathcal{M}$ is representable as the set of true atoms

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$\mathcal{G}$ is satisfiable iff $\mathcal{G}$ has a Herbrand model (over $\mathcal{L}$)

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$\mathcal{G}$ a set of universal sentences (of $\mathcal{L}$) without $=$

Definition

$$\mathrm{Gr}(\mathcal{G}) = \{\, G(t_1, \ldots, t_n) \mid \forall x_1 \cdots \forall x_n G(x_1, \ldots, x_n) \in \mathcal{G},\ t_i \text{ closed terms}\}$$

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$\mathcal{G}$ has a Herbrand model or $\mathcal{G}$ is unsatisfiable; in the latter case the following statements hold (and are equivalent):

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- all nodes in T are closed
- $\exists$ finite unsatisfiable $S \subseteq \mathrm{Gr}(\neg F)$
- by Herbrand's theorem $\neg F$ is unsatisfiable, hence F is valid ∎

Eliminating Function Symbols and Identity

Definition

1. wlog assume that in F individual and function constants occur only to the right hand of $=$
2. we replace all occurrences of $y = f(x_1, \dots, x_n)$ by $P(x_1, \dots, x_n, y)$, where P is fresh
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Lemma

F is satisfiable if and only if $F'' \wedge C(f)$ is satisfiable

Definition (Equivalence and Congruence)

- let E denote the following equivalence axioms : $\forall x\ x \fallingdotseq x \wedge \forall x \forall y\ (x \fallingdotseq y \wedge y \fallingdotseq x) \wedge \forall x \forall y \forall z\ ((x \fallingdotseq y \wedge y \fallingdotseq z) \rightarrow x \fallingdotseq z)$

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let F''' denote the result of replacing $=$ everywhere by $\fallingdotseq$

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F is satisfiable if and only if $F''' \land E \land \bigwedge_{P \in F} C(P)$ is satisfiable

Theorem

$\forall$ formula F, $\exists$ formula G not containing individual, nor function constants, nor $=$ such that $F \approx G$