Functional Programming

WS 2013/14

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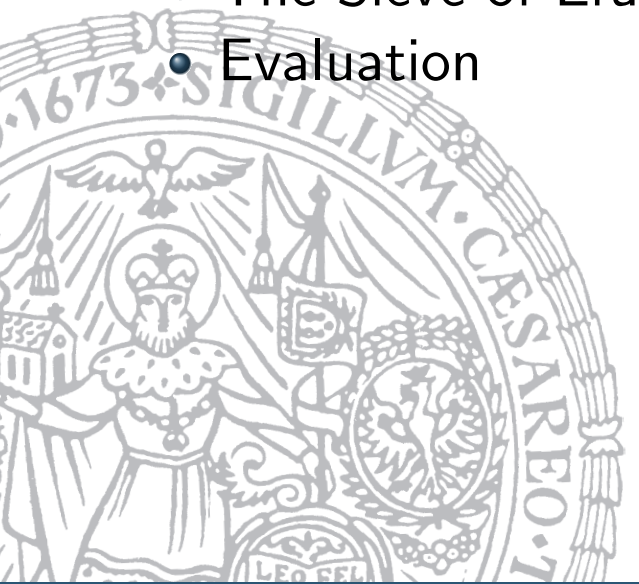
week 12

Overview



Overview

Summary of Week 13

Divide & Conquer

- solve problems recursively (until trivial problem reached)
- (easy) complexity analysis
- Master Theorem

Dynamic Programming

- recalling rather than recomputing
- save (much) time by using (slighly) more space
- efficient lookup table needed

Overview

This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction,
reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing, dynamic programming

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, monads, ...

Overview

Motivation

Idea

Only compute values that are needed for the final result.

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Only compute values that are needed for the final result.

Example

In the program

```
let rec f x = if x = 0 then 0 else f x in
Lst.hd(f 0 :: f 1 :: [])
```

the value of '`f 1`' is not needed. Nevertheless, the whole program does not terminate.

Custom Lazy Lists – 1st Iteration

Type

```
type 'a llist = Nil | Cons of ('a * 'a llist)
```

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```

Example

`Nil`	`([])`
`Cons(1,Nil)`	`([1])`
`Cons(2,Cons(1,Nil))`	`([2;1])`

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Example

```
Nil                    ([])
Cons(1,Nil)            ([1])
Cons(2,Cons(1,Nil))    ([2;1])
```

Functions

```
let hd = function Nil        -> failwith "empty list"
                | Cons(x,_) -> x

let rec from n = Cons(n,from(n+1))
```

Custom Lazy Lists – 1st Iteration (cont'd)

Problem

```
# hd(from 0);;
Stack overflow ...
```

Custom Lazy Lists – 1st Iteration (cont'd)

Problem

```
# hd(from 0);;
Stack overflow ...
```

Idea

- block computation of *tail*, until explicitly requested

Custom Lazy Lists – 1st Iteration (cont'd)

Problem

```
# hd(from 0);;
Stack overflow ...
```

Idea

- block computation of *tail*, until explicitly requested
- use `unit` function (i.e., of type `unit -> ...`)

Custom Lazy Lists – 2nd Iteration (module `UnitList`)

Type

```
type 'a llist = Nil | Cons of ('a * (unit -> 'a llist))
```

Custom Lazy Lists – 2nd Iteration (module `UnitList`)

Type

```
type 'a llist = Nil | Cons of ('a * (unit -> 'a llist))
```

Example

```
Nil                                          ([])
Cons(1, fun () -> Nil)                       ([1])
Cons(2, fun () -> Cons(1, fun () -> Nil))    ([2;1]))
```

Custom Lazy Lists – 2nd Iteration (module UnitList)

Type

```
type 'a llist = Nil | Cons of ('a * (unit -> 'a llist))
```

Example

```
Nil                                              ([])
Cons(1, fun () -> Nil)                           ([1])
Cons(2, fun () -> Cons(1, fun () -> Nil))        ([2;1]))
```

Functions

```
let hd = function Nil       -> failwith "empty list"
                | Cons(x,_) -> x

let tl = function Nil       -> failwith "empty list"
                | Cons(_,xs) -> xs ()

let rec from n = Cons(n,fun() -> from(n+1))
```

Custom Lazy Lists – 2nd Iteration (cont'd)

Now

```
# hd(from 0);;
- : int = 0

# hd(tl(from 0));;
- : int = 1
```

Custom Lazy Lists – 2nd Iteration (cont'd)

Now

```
# hd(from 0);;
- : int = 0

# hd(tl(from 0));;
- : int = 1
```

But

- strange that *tail* of `llist` is not `llist` itself

Custom Lazy Lists – 2nd Iteration (cont'd)

Now

```
# hd(from 0);;
- : int = 0

# hd(tl(from 0));;
- : int = 1
```

But

- strange that *tail* of `llist` is not `llist` itself
- use a mutually recursive type

Custom Lazy Lists – 3rd Iteration (module `UnitList`)

Type

```
type 'a cell = Nil | Cons of ('a * 'a llist)
and 'a llist = (unit -> 'a cell)
```

Custom Lazy Lists – 3rd Iteration (module `UnitList`)

Type

```
type 'a cell = Nil | Cons of ('a * 'a llist)
and 'a llist = (unit -> 'a cell)
```

Example

```
fun () -> Nil                                          ([])
fun () -> Cons(1,fun () -> Nil)                        ([1])
fun () -> Cons(2,fun () -> Cons(1,fun () -> Nil))      ([2;1])
```

Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil       -> failwith "empty"
                          | Cons(x,_) -> x
```

Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil       -> failwith "empty"
                          | Cons(x,_) -> x

let rec from n = fun() -> Cons(n,from(n+1))
```

Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil       -> failwith "empty"
                          | Cons(x,_) -> x

let rec from n = fun() -> Cons(n,from(n+1))

let rec to_list n xs = if n < 1 then [] else match xs() with
 | Nil        -> []
 | Cons(x,xs) -> x :: to_list (n-1) xs
```

Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil       -> failwith "empty"
                          | Cons(x,_) -> x

let rec from n = fun() -> Cons(n,from(n+1))

let rec to_list n xs = if n < 1 then [] else match xs() with
 | Nil        -> []
 | Cons(x,xs) -> x :: to_list (n-1) xs
```

Example

```
# from 0;;
- : int llist = <fun>

# to_list 10 (from 0);;
- : int list = [0; 1; 2; 3; 4; 5; 6; 7; 8; 9]
```

Overview

Recall

Definition (i-th Fibonacci number F_i)

$$F_i = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ F_{i-1} + F_{i-2} & \text{otherwise} \end{cases}$$

Recall

Definition (i-th Fibonacci number F_i)

$$F_i = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ F_{i-1} + F_{i-2} & \text{otherwise} \end{cases}$$

Sequence

0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 ...

Idea

Visualization

starting at 0 | 0 1

Idea

Visualization

starting at 0	0 1
starting at 1	1

Idea

Visualization

starting at 0	0 1
starting at 1	1
(+)	

Idea

Visualization

starting at 0	0 1
starting at 1	1
(+)	

Idea

Visualization

starting at 0	0 1
starting at 1	1
(+)	1

Idea

Visualization

starting at 0	0	1	1
starting at 1	1	1	
(+)	1		

Idea

Visualization

starting at 0	0	1	1
starting at 1	1	1	
(+)	1		

Idea

Visualization

starting at 0	0	1	1
starting at 1	1	1	
(+)	1	2	

Idea

Visualization

starting at 0	0	1	1	2
starting at 1	1	1	2	
$(+)$	1	2		

Idea

Visualization

starting at 0	0	1	1	2
starting at 1	1	1	2	
(+)	1	2		

Idea

Visualization

starting at 0	0	1	1	2
starting at 1	1	1	2	
(+)	1	2	3	

Idea

Visualization

starting at 0	0	1	1	2	3
starting at 1	1	1	2	3	
$(+)$	1	2	3		

Idea

Visualization

starting at 0	0	1	1	2	3
starting at 1	1	1	2	3	
(+)	1	2	3		

Idea

Visualization

starting at 0	0	1	1	2	3
starting at 1	1	1	2	3	
$(+)$	1	2	3	5	

Idea

Visualization

starting at 0	0	1	1	2	3	5
starting at 1	1	1	2	3	5	
$(+)$	1	2	3	5		

Idea

Visualization

starting at 0	0	1	1	2	3	5
starting at 1	1	1	2	3	5	
$(+)$	1	2	3	5		

Idea

Visualization

starting at 0	0	1	1	2	3	5
starting at 1	1	1	2	3	5	
$(+)$	1	2	3	5	8	

Idea

Visualization

starting at 0	0	1	1	2	3	5	8
starting at 1	1	1	2	3	5	8	
$(+)$	1	2	3	5	8		

Idea

Visualization

starting at 0	0	1	1	2	3	5	8
starting at 1	1	1	2	3	5	8	
$(+)$	1	2	3	5	8		

Idea

Visualization

starting at 0	0	1	1	2	3	5	8
starting at 1	1	1	2	3	5	8	
$(+)$	1	2	3	5	8	13	

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13
starting at 1	1	1	2	3	5	8	13	
$(+)$	1	2	3	5	8	13		

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13
starting at 1	1	1	2	3	5	8	13	
$(+)$	1	2	3	5	8	13		

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13
starting at 1	1	1	2	3	5	8	13	
$(+)$	1	2	3	5	8	13	21	

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13	21
starting at 1	1	1	2	3	5	8	13	21	
$(+)$	1	2	3	5	8	13	21		

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13	21
starting at 1	1	1	2	3	5	8	13	21	
$(+)$	1	2	3	5	8	13	21		

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Visualization

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starting at 1	1	1	2	3	5	8	13	21	
$(+)$	1	2	3	5	8	13	21	34	

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13	21	...
starting at 1	1	1	2	3	5	8	13	21	...	
$(+)$	1	2	3	5	8	13	21	34	...	

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13	21	...
starting at 1	1	1	2	3	5	8	13	21	...	
$(+)$	1	2	3	5	8	13	21	34	...	

Missing

- function to shift sequence to the left
- function to add two sequences

Idea

Visualization

starting at 0	0	1	1	2	3	5	8	13	21	...
starting at 1	1	1	2	3	5	8	13	21	...	
$(+)$	1	2	3	5	8	13	21	34	...	

Missing

- function to shift sequence to the left $\rightarrow$ `tl`
- function to add two sequences $\rightarrow$ `zip_with (+)`

Implementation (in module UnitList)

```
let tl xs = match xs() with Nil         -> failwith "empty"
                          | Cons(_,xs) -> xs

let rec zip_with f xs ys = fun() -> match (xs(),ys()) with
 | (Cons(x,xs),Cons(y,ys)) -> Cons(f x y,zip_with f xs ys)
 | _                       -> Nil

let rec fibs =
 fun() -> Cons(0,fun() -> Cons(1, zip_with (+) fibs (tl fibs)))
```

Implementation (in module UnitList)

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 | _                       -> Nil

let rec fibs =
 fun() -> Cons(0,fun() -> Cons(1, zip_with (+) fibs (tl fibs)))
```

Example

```
# to_list 10 fibs
- : int list = [0; 1; 1; 2; 3; 5; 8; 13; 21; 34]
```

Problem

Lazy Enough?

- we defer computation (i.e., call-by-name evaluation) ✓

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- we do not use memoization ✗

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Memoization

- prohibit recomputation of same expressions

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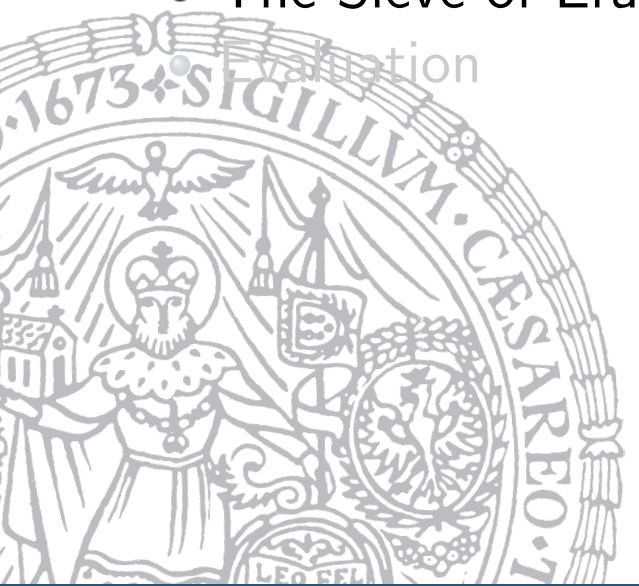
Lazy Enough?

- we defer computation (i.e., call-by-name evaluation) ✓
- we do not use memoization ✗

Memoization

- prohibit recomputation of same expressions
- built-in in OCaml's support for lazyness

Overview



Lazyness in OCaml

Keyword `lazy`

used to transform arbitrary expression into lazy expression

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Example

- `let e = lazy(print_string "test\n")`
- `let f = lazy(let rec f() = print_int 1;f() in f())`

Lazyness in OCaml

Keyword `lazy`

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Example

- `let e = lazy(print_string "test\n")`
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Function `Lazy.force`

used to evaluate lazy expressions

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Function `Lazy.force`

used to evaluate lazy expressions

Example

- `Lazy.force e`

Lazyness in OCaml

Keyword `lazy`

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Example

- `let e = lazy(print_string "test\n")`
- `let f = lazy(let rec f() = print_int 1;f() in f())`

Function `Lazy.force`

used to evaluate lazy expressions

Example

- `Lazy.force e`
- `Lazy.force f`

Lazy Lists Again (module LazyList)

Type

```
type 'a t    = 'a cell Lazy.t
and 'a cell = Nil | Cons of ('a * 'a t)
```

Lazy Lists Again (module LazyList)

Type

```
type 'a t    = 'a cell Lazy.t
and 'a cell = Nil | Cons of ('a * 'a t)
```

Example

```
lazy Nil                                        ([])
lazy (Cons(1, lazy Nil))                        ([1])
lazy (Cons(2, lazy (Cons(1, lazy Nil))))        ([2;1])
```

The Sieve of Eratosthenes (module `LazyList`)

Algorithm

start with list of all natural numbers (from 2 on)

1. mark first element h as prime
2. remove all multiples of h
3. goto step ①

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Algorithm

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Functions

```
let fc = Lazy.force
```

The Sieve of Eratosthenes (module `LazyList`)

Algorithm

start with list of all natural numbers (from 2 on)

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Functions

```
let fc = Lazy.force

let rec from n = lazy(Cons(n,from(n+1)))
```

The Sieve of Eratosthenes (module LazyList)

Algorithm

start with list of all natural numbers (from 2 on)

1. mark first element *h* as prime
2. remove all multiples of *h*
3. goto step 1

Functions

```
let fc = Lazy.force

let rec from n = lazy(Cons(n,from(n+1)))

let rec to_list n xs = if n < 1 then [] else match fc xs with
  | Nil         -> []
  | Cons(x,xs) -> x :: to_list (n-1) xs
```

The Sieve of Eratosthenes (module LazyList cont'd)

```
let rec filter p xs = lazy(match fc xs with
 | Nil         -> Nil
 | Cons(x,xs) -> if p x then Cons(x,filter p xs)
                        else fc(filter p xs)
)
```

The Sieve of Eratosthenes (module LazyList cont'd)

```
let rec filter p xs = lazy(match fc xs with
 | Nil         -> Nil
 | Cons(x,xs) -> if p x then Cons(x,filter p xs)
                        else fc(filter p xs)
)

let rec sieve xs = lazy(match fc xs with
 | Nil         -> Nil
 | Cons(x,xs) ->
  Cons(x,sieve(filter (fun y -> y mod x <> 0) xs))
)
```

The Sieve of Eratosthenes (module LazyList cont'd)

```
let rec filter p xs = lazy(match fc xs with
 | Nil         -> Nil
 | Cons(x,xs) -> if p x then Cons(x,filter p xs)
                        else fc(filter p xs)
)

let rec sieve xs = lazy(match fc xs with
 | Nil         -> Nil
 | Cons(x,xs) ->
  Cons(x,sieve(filter (fun y -> y mod x <> 0) xs))
)

let primes = sieve(from 2)
```

Overview

Evaluation (LVA Nummer: 703024-0)

Zusatzfragen

- Ich lerne aus dem Skriptum.
- Ich lerne aus den Folien.
- Ich verwende die bereitgestellten History Files.
- Die Demonstrationen im Interpreter sind hilfreich.

Anregungen für Freitext (was hat mir gefallen / nicht gefallen)

- Vortragsstil (Folienvortrag, Demonstrationen im Interpreter)
- Tafelbild (Schrift)
- Einsatz von Tools (lips, lambda tree tool)
- Mischung Theorie/Programmierung
- Programmierwettbewerb (PCP Competition): Warum (nicht) teilgenommen?