Functional Programming

WS 2014/15

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week 05

Overview

- Week 5 - λ-Calculus
  - Summary of Week 4
  - λ-Calculus - Introduction
  - λ-Calculus - Formalities
  - The λ Interpreter `lips`

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Summary of Week 4

Binary Trees

- at most 2 children per node
- applications
  - search trees
  - Huffman coding

Huffman Coding

- Idea: use shortest codewords for most frequent symbols
- Application: (lossless) data compression

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This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction,
reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing, dynamic programming

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, monads, ...

Overview

Origin

Goal

- find a framework in which every algorithm can be defined
- universal language

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Result

- Turing machines (Turing, 1930s)
- λ-Calculus (Church, 1930s)
- ...

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- find a framework in which every algorithm can be defined
- universal language

Result

- Turing machines (Turing, 1930s)
- λ-Calculus (Church, 1930s)
- ...

Syntax

λ-Terms

$$t ::= \quad x \quad | \; (\lambda x.t) \; | \; (t \;\; t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Syntax

λ-Terms

$$t ::= \overbrace{x}^{\text{Variable}} \mid (\lambda x.t) \mid (t\ \ t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Syntax

λ-Terms

$$t ::= \quad x \quad \mid \underbrace{(\lambda x.t)}_{\color{red}\text{Abstraction}} \mid (t \;\; t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Syntax

λ-Terms

$$t ::= x \mid (\lambda x.t) \mid \overbrace{(t\ t)}^{\text{Application}}$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Syntax

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid (t \ t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Syntax

λ-Terms

$$t ::= \; x \; \mid (\lambda x.t) \mid (t \; t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Conventions

$$(\lambda x.x)$$
$$(\lambda x.(\lambda y.x))$$
$$(\lambda x.(\lambda y.(\lambda z.((x \; z) \; (y \; z)))))$$
$$(\lambda x.((\lambda y.(\lambda z.(z \; y))) \; x))$$

Syntax

λ-Terms

$$t ::= \quad x \quad | \; (\lambda x.t) \; | \; (t \;\; t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Conventions (omit outermost parentheses)

$$\lambda x.x$$
$$\lambda x.(\lambda y.x)$$
$$\lambda x.(\lambda y.(\lambda z.((x \; z) \; (y \; z))))$$
$$\lambda x.((\lambda y.(\lambda z.(z \; y))) \; x)$$

Syntax

λ-Terms

$$t ::= \quad x \quad | \; (\lambda x.t) \; | \; (t \;\; t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Conventions (combine nested lambdas)

$$\lambda x.x$$
$$\lambda xy.x$$
$$\lambda xyz.((x \;\; z) \;\; (y \;\; z))$$
$$\lambda x.((\lambda yz.(z \;\; y)) \;\; x)$$

Syntax

λ-Terms

$$t ::= \quad x \quad | \; (\lambda x.t) \; | \; (t \;\; t)$$

$\mathcal{T}(\mathcal{V})$ set of all λ-terms over set of variables $\mathcal{V}$

Conventions (application is left-associative and binds strongest)

$$\lambda x.x$$
$$\lambda xy.x$$
$$\lambda xyz.x \; z \; (y \;\; z)$$
$$\lambda x.(\lambda yz.z \;\; y) \;\; x$$

Intuition

Example

λ-terms

- $\lambda x.\text{add}\ x\ \overline{1}$
- $(\lambda x.\text{add}\ x\ \overline{1})\ \overline{2}$
- $\text{if}\ \text{true}\ \overline{1}\ \overline{0}$
- $\text{pair}\ \overline{2}\ \overline{4}$
- $\text{fst}(\text{pair}\ \overline{2}\ \overline{4})$
- $\lambda xy.\text{add}\ x\ y$
- $\lambda x.(\lambda y.\text{add}\ x\ y)$

Intuition

Example

λ-terms OCaml

- $\lambda x.\mathsf{add}\ x\ \overline{1}$
- $(\lambda x.\mathsf{add}\ x\ \overline{1})\ \overline{2}$
- $\mathsf{if}\ \mathsf{true}\ \overline{1}\ \overline{0}$
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Intuition

Example

λ-terms

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OCaml

- `fun x -> x+1`

Intuition

Example

λ-terms

- $\lambda x.\mathsf{add}\ x\ \overline{1}$
- $(\lambda x.\mathsf{add}\ x\ \overline{1})\ \overline{2}$
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- $\mathsf{fst}(\mathsf{pair}\ \overline{2}\ \overline{4})$
- $\lambda xy.\mathsf{add}\ x\ y$
- $\lambda x.(\lambda y.\mathsf{add}\ x\ y)$

OCaml

- `fun x -> x+1`
- `(fun x -> x+1) 2` $\rightarrow^{+}$ 3

Intuition

Example

λ-terms

- $\lambda x.\mathsf{add}\ x\ \overline{1}$
- $(\lambda x.\mathsf{add}\ x\ \overline{1})\ \overline{2}$
- $\mathsf{if}\ \mathsf{true}\ \overline{1}\ \overline{0}$
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- $\lambda xy.\mathsf{add}\ x\ y$
- $\lambda x.(\lambda y.\mathsf{add}\ x\ y)$

OCaml

- `fun x -> x+1`
- `(fun x -> x+1) 2` $\to^+$ `3`
- `if true then 1 else 0` $\to$ `1`

Intuition

Example

λ-terms

- $\lambda x.\text{add}\ x\ \overline{1}$
- $(\lambda x.\text{add}\ x\ \overline{1})\ \overline{2}$
- $\text{if true}\ \overline{1}\ \overline{0}$
- $\text{pair}\ \overline{2}\ \overline{4}$
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- $\lambda xy.\text{add}\ x\ y$
- $\lambda x.(\lambda y.\text{add}\ x\ y)$

OCaml

- `fun x -> x+1`
- `(fun x -> x+1) 2` $\to^+$ `3`
- `if true then 1 else 0` $\to$ `1`
- `(2,4)`

Intuition

Example

λ-terms

- $\lambda x.\mathsf{add}\ x\ \overline{1}$
- $(\lambda x.\mathsf{add}\ x\ \overline{1})\ \overline{2}$
- $\mathsf{if\ true}\ \overline{1}\ \overline{0}$
- $\mathsf{pair}\ \overline{2}\ \overline{4}$
- $\mathsf{fst}(\mathsf{pair}\ \overline{2}\ \overline{4})$
- $\lambda xy.\mathsf{add}\ x\ y$
- $\lambda x.(\lambda y.\mathsf{add}\ x\ y)$

OCaml

- `fun x -> x+1`
- `(fun x -> x+1) 2` $\to^+$ `3`
- `if true then 1 else 0` $\to$ `1`
- `(2,4)`
- `fst(2,4)` $\to$ `2`

Intuition

Example

λ-terms

- $\lambda x.\mathsf{add}\ x\ \overline{1}$
- $(\lambda x.\mathsf{add}\ x\ \overline{1})\ \overline{2}$
- $\mathsf{if\ true}\ \overline{1}\ \overline{0}$
- $\mathsf{pair}\ \overline{2}\ \overline{4}$
- $\mathsf{fst}(\mathsf{pair}\ \overline{2}\ \overline{4})$
- $\lambda xy.\mathsf{add}\ x\ y$
- $\lambda x.(\lambda y.\mathsf{add}\ x\ y)$

OCaml

- `fun x -> x+1`
- `(fun x -> x+1) 2` $\to^+$ `3`
- `if true then 1 else 0` $\to$ `1`
- `(2,4)`
- `fst(2,4)` $\to$ `2`
- `fun x y -> x + y`

Intuition

Example

λ-terms

- $\lambda x.\mathsf{add}\ x\ \overline{1}$
- $(\lambda x.\mathsf{add}\ x\ \overline{1})\ \overline{2}$
- $\mathsf{if}\ \mathsf{true}\ \overline{1}\ \overline{0}$
- $\mathsf{pair}\ \overline{2}\ \overline{4}$
- $\mathsf{fst}(\mathsf{pair}\ \overline{2}\ \overline{4})$
- $\lambda xy.\mathsf{add}\ x\ y$
- $\lambda x.(\lambda y.\mathsf{add}\ x\ y)$

OCaml

- `fun x -> x+1`
- `(fun x -> x+1) 2` $\to^+$ `3`
- `if true then 1 else 0` $\to$ `1`
- `(2,4)`
- `fst(2,4)` $\to$ `2`
- `fun x y -> x + y`
- `fun x -> fun y -> x + y`

Intuition

Example

λ-terms

- $\lambda x.\mathsf{add}\ x\ \overline{1}$
- $(\lambda x.\mathsf{add}\ x\ \overline{1})\ \overline{2}$
- if true $\overline{1}\ \overline{0}$
- pair $\overline{2}\ \overline{4}$
- fst(pair $\overline{2}\ \overline{4}$)
- $\lambda xy.\mathsf{add}\ x\ y$
- $\lambda x.(\lambda y.\mathsf{add}\ x\ y)$

OCaml

- `fun x -> x+1`
- `(fun x -> x+1) 2` $\to^+$ 3
- `if true then 1 else 0` $\to$ 1
- `(2,4)`
- `fst(2,4)` $\to$ 2
- `fun x y -> x + y`
- `fun x -> fun y -> x + y`

Remark

'$\overline{0}$', '$\overline{1}$', '$\overline{2}$', '$\overline{3}$', '$\overline{4}$', 'add', 'fst', 'if', 'pair', and 'true' are just abbreviations for more complex λ-terms

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Subterms

Definition

$\mathcal{S}\mathrm{ub}(t)$ is set of subterms of t

$$\mathcal{S}\mathrm{ub}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} & t = x \\ \{t\} \cup \mathcal{S}\mathrm{ub}(u) & t = \lambda x.u \\ \{t\} \cup \mathcal{S}\mathrm{ub}(u) \cup \mathcal{S}\mathrm{ub}(v) & t = u\ v \end{cases}$$

Subterms

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Example

$$\mathcal{S}\mathrm{ub}(\lambda xy.x)$$

Subterms

Definition

$\mathcal{S}\mathsf{ub}(t)$ is set of subterms of t

$$\mathcal{S}\mathsf{ub}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} & t = x \\ \{t\} \cup \mathcal{S}\mathsf{ub}(u) & t = \lambda x.u \\ \{t\} \cup \mathcal{S}\mathsf{ub}(u) \cup \mathcal{S}\mathsf{ub}(v) & t = u\ v \end{cases}$$

Example

$$\mathcal{S}\mathsf{ub}(\lambda xy.x) = \{\lambda xy.x\} \cup \mathcal{S}\mathsf{ub}(\lambda y.x)$$

Subterms

Definition

$\mathcal{S}\text{ub}(t)$ is set of subterms of t

$$\mathcal{S}\text{ub}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} & t = x \\ \{t\} \cup \mathcal{S}\text{ub}(u) & t = \lambda x.u \\ \{t\} \cup \mathcal{S}\text{ub}(u) \cup \mathcal{S}\text{ub}(v) & t = u\ v \end{cases}$$

Example

$$\begin{aligned} \mathcal{S}\text{ub}(\lambda xy.x) &= \{\lambda xy.x\} \cup \mathcal{S}\text{ub}(\lambda y.x) \\ &= \{\lambda xy.x, \lambda y.x\} \cup \mathcal{S}\text{ub}(x) \end{aligned}$$

Subterms

Definition

$\mathcal{S}\mathsf{ub}(t)$ is set of subterms of t

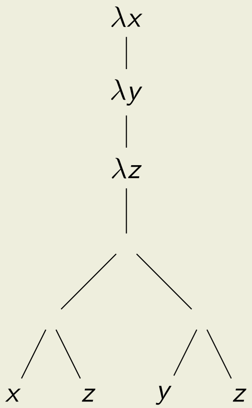
$$\mathcal{S}\mathsf{ub}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} & t = x \\ \{t\} \cup \mathcal{S}\mathsf{ub}(u) & t = \lambda x.u \\ \{t\} \cup \mathcal{S}\mathsf{ub}(u) \cup \mathcal{S}\mathsf{ub}(v) & t = u\ v \end{cases}$$

Example

$$\begin{aligned} \mathcal{S}\mathsf{ub}(\lambda xy.x) &= \{\lambda xy.x\} \cup \mathcal{S}\mathsf{ub}(\lambda y.x) \\ &= \{\lambda xy.x, \lambda y.x\} \cup \mathcal{S}\mathsf{ub}(x) \\ &= \{\lambda xy.x, \lambda y.x, x\} \end{aligned}$$

Syntax Trees

Example

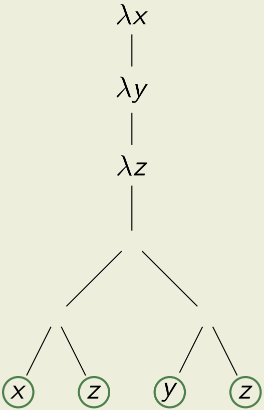


$$t = \lambda xyz.x\ z\ (y\ z)$$

$$\mathcal{S}\mathrm{ub}(t) = \{t, \lambda yz.x\ z\ (y\ z),\ \lambda z.x\ z\ (y\ z),\ x\ z\ (y\ z), x\ z, y\ z,\ x, z, y\}$$

Syntax Trees

Example



$t = \lambda xyz.x\ z\ (y\ z)$

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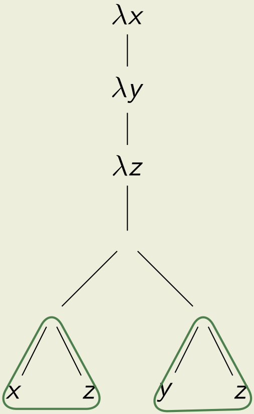
Syntax Trees

Example



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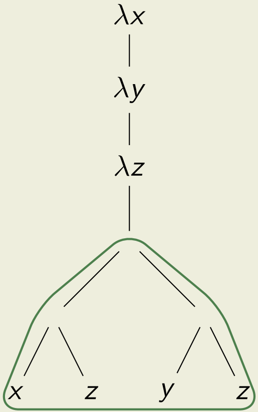
Syntax Trees

Example



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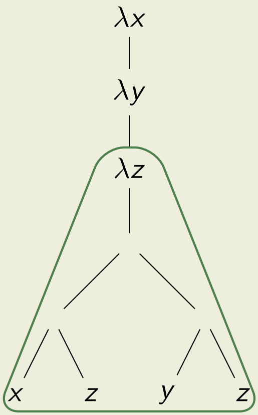
Syntax Trees

Example



$$t = \lambda xyz.x\ z\ (y\ z)$$

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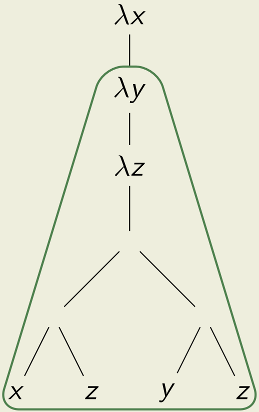
Syntax Trees

Example



$$t = \lambda xyz.x\ z\ (y\ z)$$

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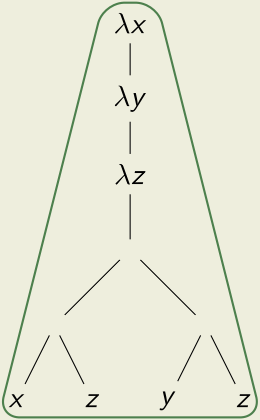
Syntax Trees

Example



$$t = \lambda xyz.x\ z\ (y\ z)$$

$$\mathcal{S}\text{ub}(t) = \{t, \lambda yz.x\ z\ (y\ z), \lambda z.x\ z\ (y\ z), x\ z\ (y\ z), x\ z, y\ z, x, z, y\}$$

Variables

Definition

variables

$$\mathcal{V}\text{ar}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} & t = x \\ \{x\} \cup \mathcal{V}\text{ar}(u) & t = \lambda x.u \\ \mathcal{V}\text{ar}(u) \cup \mathcal{V}\text{ar}(v) & t = u\ v \end{cases}$$

Free and Bound Variables

Definition

free variables

$$\mathcal{FV}\text{ar}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} & t = x \\ \mathcal{FV}\text{ar}(u) \setminus \{x\} & t = \lambda x.u \\ \mathcal{FV}\text{ar}(u) \cup \mathcal{FV}\text{ar}(v) & t = u\ v \end{cases}$$

Free and Bound Variables

Definition

free variables

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bound variables

$$\mathcal{BV}\mathrm{ar}(t) \stackrel{\text{def}}{=} \begin{cases} \varnothing & t = x \\ \{x\} \cup \mathcal{BV}\mathrm{ar}(u) & t = \lambda x.u \\ \mathcal{BV}\mathrm{ar}(u) \cup \mathcal{BV}\mathrm{ar}(v) & t = u\ v \end{cases}$$

Free and Bound Variables

Definition

free variables

$$\mathcal{FV}\text{ar}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} & t = x \\ \mathcal{FV}\text{ar}(u) \setminus \{x\} & t = \lambda x.u \\ \mathcal{FV}\text{ar}(u) \cup \mathcal{FV}\text{ar}(v) & t = u\ v \end{cases}$$

bound variables

$$\mathcal{BV}\text{ar}(t) \stackrel{\text{def}}{=} \begin{cases} \varnothing & t = x \\ \{x\} \cup \mathcal{BV}\text{ar}(u) & t = \lambda x.u \\ \mathcal{BV}\text{ar}(u) \cup \mathcal{BV}\text{ar}(v) & t = u\ v \end{cases}$$

A λ-term without free variables is called closed.

Examples

t	$\mathcal{V}\mathsf{ar}(t)$	$\mathcal{FV}\mathsf{ar}(t)$	$\mathcal{BV}\mathsf{ar}(t)$	closed
$\lambda x.x$				
$x\ y$				
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\mathsf{ar}(t)$	$\mathcal{F}\mathcal{V}\mathsf{ar}(t)$	$\mathcal{B}\mathcal{V}\mathsf{ar}(t)$	closed
$\lambda x.x$	$\{x\}$			
$x\ y$				
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\mathsf{ar}(t)$	$\mathcal{F}\mathcal{V}\mathsf{ar}(t)$	$\mathcal{B}\mathcal{V}\mathsf{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$		
$x\ y$				
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\mathsf{ar}(t)$	$\mathcal{FV}\mathsf{ar}(t)$	$\mathcal{BV}\mathsf{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	
$x\ y$				
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\mathsf{ar}(t)$	$\mathcal{F}\mathcal{V}\mathsf{ar}(t)$	$\mathcal{B}\mathcal{V}\mathsf{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$				
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\mathrm{ar}(t)$	$\mathcal{F}\mathcal{V}\mathrm{ar}(t)$	$\mathcal{B}\mathcal{V}\mathrm{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$			
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\text{ar}(t)$	$\mathcal{FV}\text{ar}(t)$	$\mathcal{BV}\text{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$	$\{x, y\}$		
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\text{ar}(t)$	$\mathcal{FV}\text{ar}(t)$	$\mathcal{BV}\text{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$	$\{x, y\}$	$\varnothing$	
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\mathrm{ar}(t)$	$\mathcal{F}\mathcal{V}\mathrm{ar}(t)$	$\mathcal{B}\mathcal{V}\mathrm{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$	$\{x, y\}$	$\varnothing$	✗
$(\lambda x.x)\ x$				
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\text{ar}(t)$	$\mathcal{FV}\text{ar}(t)$	$\mathcal{BV}\text{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$	$\{x, y\}$	$\varnothing$	✗
$(\lambda x.x)\ x$	$\{x\}$			
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\mathrm{ar}(t)$	$\mathcal{F}\mathcal{V}\mathrm{ar}(t)$	$\mathcal{B}\mathcal{V}\mathrm{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$	$\{x, y\}$	$\varnothing$	✗
$(\lambda x.x)\ x$	$\{x\}$	$\{x\}$		
$\lambda x.x\ y\ z$				

Examples

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$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$	$\{x, y\}$	$\varnothing$	✗
$(\lambda x.x)\ x$	$\{x\}$	$\{x\}$	$\{x\}$	
$\lambda x.x\ y\ z$				

Examples

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$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x,y\}$	$\{x,y\}$	$\varnothing$	✗
$(\lambda x.x)\ x$	$\{x\}$	$\{x\}$	$\{x\}$	✗
$\lambda x.x\ y\ z$				

Examples

t	$\mathcal{V}\text{ar}(t)$	$\mathcal{FV}\text{ar}(t)$	$\mathcal{BV}\text{ar}(t)$	closed
$\lambda x.x$	$\{x\}$	$\varnothing$	$\{x\}$	✓
$x\ y$	$\{x, y\}$	$\{x, y\}$	$\varnothing$	✗
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$\lambda x.x\ y\ z$	$\{x, y, z\}$			

Examples

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Solution

rename bound variables where necessary

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let y = 3 and z = 2;;
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Substitutions

Definition

function from variables to terms

$$\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{V})$$

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$$\sigma = \{x/\lambda x.x\} \text{ hence } \sigma(x) = \lambda x.x \text{ and } \sigma(y) = y$$

Substitutions (cont'd)

Definition (Application)

apply substitution $\sigma = \{x/s\}$ to term t

$$t\sigma \stackrel{\text{def}}{=} \begin{cases} s & t = x \\ y & t = y,\ x \neq y \\ (u\sigma)\ (v\sigma) & t = u\ v \\ \lambda x.u & t = \lambda x.u \\ \lambda y.(u\sigma) & t = \lambda y.u,\ x \neq y,\ y \notin \mathcal{FV}\mathrm{ar}(s) \\ \lambda y'.((u\{y/y'\})\sigma) & t = \lambda y.u,\ x \neq y,\ y \in \mathcal{FV}\mathrm{ar}(s),\ y' \text{ fresh} \end{cases}$$

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Examples

$$(\lambda x.x)\ (\lambda x.x) \to_\beta$$

$$(\lambda xy.y)\ (\lambda x.x) \to_\beta$$

$$(\lambda xyz.x\ z\ (y\ z))\ (\lambda x.x) \to_\beta$$

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$$\lambda x.x \to_\beta$$

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Examples

$$
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(\lambda x.x)\ (\lambda x.x) &\to_\beta \lambda x.x \\
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β-Reduction

Definition (Context)

context $C \in \mathcal{C}(\mathcal{V})$

$$C ::= \Box \mid \lambda x.C \mid C\ t \mid t\ C$$

with $\Box \notin \mathcal{V}$, $x \in \mathcal{V}$ and $t \in \mathcal{T}(\mathcal{V})$

- $C[s]$ denotes replacing $\Box$ by term s in context C

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Example

$$C_1 = \Box \qquad C_1[\lambda x.x] =$$
$$C_2 = x\ \Box \qquad C_2[\lambda x.x] =$$
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β-Reduction (cont'd)

Definition (β-step)

if exist context C and terms s, u, and v such that

$$s = C[(\lambda x.u)\ v]$$

then

$$s \quad \to_\beta \quad C[u\{x/v\}]$$

is a β-step with redex $(\lambda x.u)\ v$ and contractum $u\{x/v\}$

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- $s \to_\beta^* t$ is sequence with $n \geq 0$ (s β-reduces to t)

β-Reduction

Example

$$\Omega = (\lambda x.x\ x)\ (\lambda x.x\ x)$$
$$K_* = \lambda xy.y$$
$$I_2 = \lambda xy.x\ y$$

$K_*\ \Omega$

$K_*\ \Omega$

$I_2\ I_2$

β-Reduction

Example

$$\Omega = (\lambda x.x\ x)\ (\lambda x.x\ x)$$
$$K_* = \lambda xy.y$$
$$I_2 = \lambda xy.x\ y$$

$K_*\ \Omega \to_\beta K_*\ \Omega$

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What Are the Results of Computations?

Idea

- only terms in λ-calculus
- express functions and values through λ-terms

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$$\lambda x.x$$
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$$(\lambda x.x)\ y$$

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Example

$$\begin{array}{rl} \lambda x.x & \text{NF} \\ (\lambda x.x)\ y & \text{not NF} \end{array}$$

Overview

- Week 5 - λ-Calculus
  - Summary of Week 4
  - λ-Calculus - Introduction
  - λ-Calculus - Formalities
  - The λ Interpreter `lips`

Lambda Interpreter for Pure Students

developed by Michael Brunner (bachelor thesis)

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λ-Terms

$$t ::= \mathtt{x} \mid (\backslash\mathtt{x}\,.\,t) \mid (t\ t)$$

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Conventions

- interpreter command `!pretty` toggles use of conventions for printing
- nested abstractions use spaces to separate variable names, e.g.,

$\lambda xy.x$	`\x y.x`
$\lambda x_1.y$	`\x1.y`

Result

Normal Forms

- result of input is corresponding NF
  - ```
    > (\x.x) (\x.x)
    NF: (\x.x)
    ```
    ```

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  - `> (\x.x) (\x.x)`
    `NF: (\x.x)`

Evaluation Strategy

- `!by_value` activates call-by-value evaluation (next lecture)
- `!by_name` activates call-by-name evaluation (next lecture)
- `!trace` toggles tracing

Abbreviations & Initialisation

Interpreter Command

$$\texttt{!def}\ \langle name \rangle = t$$

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```
> !def I = \x.x
> !def K = \x y.x
> !def S = \x y z.x z (y z)
> S K I
NF: \z.z
```

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$$\texttt{!def}\ \langle name \rangle = t$$

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```
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> S K I
NF: \z.z
```

`.lambdainit`

content of file `.lambdainit` is loaded on start-up of `lips`