Automated Theorem Proving

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Summary Last Lecture

Definition

$$\frac{C \vee A \quad D \vee \neg B}{(C \vee D)\sigma_1} \qquad \frac{C \vee A \vee B}{(C \vee A)\sigma_1}$$

$$\frac{C \vee s \neq s'}{C\sigma_2} \qquad \frac{C \vee s = t \quad D \vee L[s']}{(C \vee D \vee L[t])\sigma_2}$$

- same conditions on σ_1, σ_2 as before
- $A\sigma_1$ is strictly maximal with respect to $C\sigma_1$; $\neg B\sigma_1$ is maximal with respect to $D\sigma_1$
- the equation $(s = t)\sigma_2$ and the literal $L[s']\sigma_2$ are maximal with respect to $D\sigma_2$

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Theorem

ordered paramodulation is sound and complete

Definition

equations $\mathcal{E}$ are ground complete wrt $\succ$ if $\mathcal{E}^{\succ}$ is complete on ground terms

Definition (superposition with equations)

$$\frac{s = t \quad w[u] = v}{(w[t] = v)\sigma}$$

- σ is mgu of s and u; $t\sigma \not\succeq s\sigma$, $v\sigma \not\succeq w[u]\sigma$ and u is not a variable
- $(w[t] = v)\sigma$ is an ordered critical pair

Theorem

$\succ$ a complete reduction order; a set of equations E is ground complete wrt $\succ$ iff $\forall$ ordered critical pairs $(w[t] = v)\sigma$ (with overlapping term $w[u]\sigma$) and $\forall$ ground substitutions τ: if $w[u]\sigma\tau \succ w[t]\sigma\tau$ and $w[u]\sigma\tau \succ v\sigma\tau$ then $w[t]\sigma\tau \downarrow v\sigma\tau$

Outline of the Lecture

Early Approaches in Automated Reasoning

Herbrand's theorem for dummies, Gilmore's prover, method of Davis and Putnam

Starting Points

resolution, tableau provers, Skolemisation, ordered resolution, redundancy and deletion

Automated Reasoning with Equality

paramodulation, ordered completion and proof orders, superposition

Applications of Automated Reasoning

Neuman-Stubblebinde Key Exchange Protocol, Robbins problem

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Ordered Completion

deduction $\quad \mathcal{E};\mathcal{R} \vdash \mathcal{E} \cup \{s = t\};\mathcal{R}$

if $s \leftrightarrow_{\mathcal{E}\cup\mathcal{R}} w \leftrightarrow_{\mathcal{E}\cup\mathcal{R}} t$, $s \not\succ w$, $t \not\succ w$

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orientation	$\mathcal{E} \cup \{s = t\};\mathcal{R} \vdash \mathcal{E};\mathcal{R} \cup \{s \to t\}$	if $s \succ t$

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- its limit is $(\mathcal{E}_\infty;\mathcal{R}_\infty)$; here $\mathcal{E}_\infty = \bigcup_{i\geqslant 0}\bigcap_{j\geqslant i}\mathcal{E}_j$; $\mathcal{R}_\infty = \bigcup_{i\geqslant 0}\bigcap_{j\geqslant i}\mathcal{R}_j$

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$$s = s_0 \ \rho_0 \ s_1 \ \rho_1 \ s_2 \cdots s_{n-1} \ \rho_{n-1} \ s_n = t \qquad n \geqslant 0$$

1. $(s_i \ \rho_i \ s_{i+1}) = (w[u\sigma] \leftrightarrow w[v\sigma])$ with $u = v \in \mathcal{E}$
2. $(s_i \ \rho_i \ s_{i+1}) = (w[u\sigma] \rightarrow w[v\sigma])$ with $u \rightarrow v \in \mathcal{E}^{\succ} \cup \mathcal{R}$
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Fact

1. $\exists$ *rewrite proof iff the equations are joinable wrt* $\mathcal{R} \cup \mathcal{E}^{\succ}$
2. *whenever* $\mathcal{E};\mathcal{R} \vdash \mathcal{E}';\mathcal{R}'$ *then the same equations are provable in* $\mathcal{E};\mathcal{R}$ *as in* $\mathcal{E}';\mathcal{R}'$; *however proofs may become simpler*

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cost measure of proof steps

$$\text{cost of } s[u] \ \rho \ s[v] = \begin{cases} (\{s[u]\}, u, \rho, s[v]) & \text{if } s[u] \succ s[v] \\ (\{s[v]\}, v, \rho, s[u]) & \text{if } s[v] \succ s[u] \\ (\{s[u], s[v]\}, \bot, \bot, \bot) & \text{otherwise} \end{cases}$$

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$\bot$ is supposed to be minimal in all orders; let $\succ_\pi$ the multiset extension of the cost measure; then $\succ_\pi$ denotes a well-founded order on proofs

Fact

each completion step decreases the cost of certain proofs

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Definition

a derivation is fair if each ordered critical pair $u = v \in \mathcal{E}_\infty \cup \mathcal{R}_\infty$ is an element of some $\mathcal{E}_i$

Theorem

let $(\mathcal{E}_0; \mathcal{R}_0), (\mathcal{E}_1; \mathcal{R}_1), \dots$ *be a fair ordered completion derivation with* $\mathcal{R}_0 = \varnothing$*; then the following is equivalent:*

1. $s = t$ *is a consequence of* $\mathcal{E}_0$

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2. $s = t$ *has a rewrite proof in* $\mathcal{E}_\infty^{\succ} \cup \mathcal{R}_\infty$

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Definitions

- let $\mathcal{E}$ be a set of equations and $s = t$ an equation (possibly containing variables); then $\mathcal{E} \models s = t$ is the word problem for $\mathcal{E}$
- the word problem becomes a refutation theorem proving problem once we consider the clause form of the negation of the word problem:
  1. a set of positive unit equations in $\mathcal{E}$
  2. a ground disequation obtained by negation and Skolemisation of $s = t$

Completeness of Superposition

Corollary

superposition with equations is sound and complete, that is, if $\mathcal{C}$ is the clause representation of the (negated) word problem $\mathcal{E} \models s = t$, then the saturation of $\mathcal{C}$ wrt to superposition (and equality resolution) contains $\Box$ iff $\mathcal{E} \models s = t$

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