Functional Programming
Exercises Week 8
(for December 4, 2015)

1. Consider the following data type, representing first-order terms:

```
type term = Var of string | Fun of string * term list
```

The term $f(x, y)$, for example, corresponds to `Fun ("f", [Var "x", Var "y"])`. Implement a function `vars : term -> string list` that computes the list of all variables occurring in a given term.

2. Substitutions are functions from variables to terms (type `string -> term` in OCaml). Define the empty substitution `id : string -> term` (i.e., a substitution that does not change anything when applied to a term) and a function `mk_subst : string -> term -> (string -> term)` where `mk_subst` x t creates the singleton substitution that maps x to t and leaves all other variables unchanged (for brevity, this is written $\{x/t\}$).

3. Implement a function `subst : (string -> term) -> term -> term` that applies a substitution to a term. Example: `subst` $\{x/t\}\ f(x, y) = f(t, y)$.

4. Use structural induction on `term`s to show that whenever two substitutions σ and τ coincide on all variables occurring in a term t, then applying σ to t and τ to t both result in the same term, i.e., $(\forall x \in \texttt{vars}\ t.\, \sigma\ x = \tau\ x) \implies \texttt{subst}\ \sigma\ t = \texttt{subst}\ \tau\ t$.

5. Use structural induction on `list`s to show that whenever $fx = x$ for all $x \in xs$, then `map` $f\ xs = xs$, i.e., $(\forall x \in xs.\, f\ x = x) \implies \texttt{map}\ f\ xs = xs$.

6. ($\star$) Use the result of Exercise 5 to prove (by structural induction on `term`s) that substituting for a variable that does not occur in a term, does not change the term, i.e., $x \notin \texttt{vars}\ t \implies \texttt{subst}\ \{x/s\}\ t = t$.

7. ($\star$) Use the result of Exercise 6 to prove (by structural induction on `term`s) that the order of applying singleton substitutions is irrelevant whenever they "do not interfere." More formally:

$$x \neq y \land x \notin \texttt{vars}\ s \land y \notin \texttt{vars}\ u \implies$$
$$\texttt{subst}\ \{y/s\}\ (\texttt{subst}\ \{x/u\}\ t) = \texttt{subst}\ \{x/u\}\ (\texttt{subst}\ \{y/x\}\ t)$$