Functional Programming

WS 2015/16

Cezary Kaliszyk (VO+PS)
Yann Savoye (PS)
Łukasz Czajka (PS)

Computational Logic
Institute of Computer Science
University of Innsbruck

week 06

Overview

- Week 6 - λ Calculus, Evaluation Strategies
  - Summary of Week 5
  - λ-Calculus - Data Types
  - Evaluation Strategies

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  - Summary of Week 5
  - λ-Calculus - Data Types
  - Evaluation Strategies

λ-Calculus

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid (t \;\; t)$$

λ-Calculus

λ-Terms

$$t ::= \overbrace{x}^{\text{Variable}} \mid (\lambda x.t) \mid (t\ t)$$

λ-Calculus

λ-Terms

$$t ::= \quad x \quad | \underbrace{(\lambda x.t)}_{\text{Abstraction}} | \; (t \;\; t)$$

λ-Calculus

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid \overbrace{(t\ \ t)}^{\text{Application}}$$

λ-Calculus

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid (t \;\; t)$$

Example

$$x \;\; y$$
$$\lambda x.x$$
$$\lambda xy.x$$
$$\lambda x.x \;\; x$$
$$(\lambda x.x) \;\; x$$

λ-Calculus

λ-Terms

$$t ::= x \mid (\lambda x.t) \mid (t\ t)$$

Example

$x\ y$ $\quad (x\ y)$

$\lambda x.x$

$\lambda xy.x$

$\lambda x.x\ x$

$(\lambda x.x)\ x$

λ-Calculus

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid (t \; t)$$

Example

$x \; y$	$(x \; y)$	"*x applied to y*"
$\lambda x.x$		
$\lambda xy.x$		
$\lambda x.x \; x$		
$(\lambda x.x) \; x$		

λ-Calculus

λ-Terms

$$t ::= \quad x \quad | \, (\lambda x.t) \, | \, (t \;\; t)$$

Example

$x \; y$	$(x \; y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	
$\lambda xy.x$		
$\lambda x.x \; x$		
$(\lambda x.x) \; x$		

λ-Calculus

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid (t \ \ t)$$

Example

$x \ y$	$(x \ y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	*"lambda x to x"*
$\lambda xy.x$		
$\lambda x.x \ x$		
$(\lambda x.x) \ x$		

λ-Calculus

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid (t \ t)$$

Example

$x\ y$	$(x\ y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	*"lambda x to x"*
$\lambda xy.x$	$(\lambda x.(\lambda y.x))$	
$\lambda x.x\ x$		
$(\lambda x.x)\ x$		

λ-Calculus

λ-Terms

$$t ::= \quad x \quad | \, (\lambda x.t) \, | \, (t \; t)$$

Example

$x \; y$	$(x \; y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	*"lambda x to x"*
$\lambda xy.x$	$(\lambda x.(\lambda y.x))$	*"lambda x y to x"*
$\lambda x.x \; x$		
$(\lambda x.x) \; x$		

λ-Calculus

λ-Terms

$$t ::= \quad x \quad | \, (\lambda x.t) \, | \, (t \;\; t)$$

Example

$x \; y$	$(x \; y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	*"lambda x to x"*
$\lambda xy.x$	$(\lambda x.(\lambda y.x))$	*"lambda x y to x"*
$\lambda x.x \; x$	$(\lambda x.(x \; x))$	
$(\lambda x.x) \; x$		

λ-Calculus

λ-Terms

$$t ::= \quad x \quad | \; (\lambda x.t) \; | \; (t \;\; t)$$

Example

$x \; y$	$(x \; y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	*"lambda x to x"*
$\lambda xy.x$	$(\lambda x.(\lambda y.x))$	*"lambda x y to x"*
$\lambda x.x \; x$	$(\lambda x.(x \; x))$	*"lambda x to (x applied to x)"*
$(\lambda x.x) \; x$		

λ-Calculus

λ-Terms

$$t ::= \quad x \quad \mid (\lambda x.t) \mid (t \ \ t)$$

Example

$x\ y$	$(x\ y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	*"lambda x to x"*
$\lambda xy.x$	$(\lambda x.(\lambda y.x))$	*"lambda x y to x"*
$\lambda x.x\ x$	$(\lambda x.(x\ x))$	*"lambda x to (x applied to x)"*
$(\lambda x.x)\ x$	$((\lambda x.x)\ x)$	

λ-Calculus

λ-Terms

$$t ::= \quad x \quad | \, (\lambda x.t) \, | \, (t \;\; t)$$

Example

$x \; y$	$(x \; y)$	*"x applied to y"*
$\lambda x.x$	$(\lambda x.x)$	*"lambda x to x"*
$\lambda xy.x$	$(\lambda x.(\lambda y.x))$	*"lambda x y to x"*
$\lambda x.x \; x$	$(\lambda x.(x \; x))$	*"lambda x to (x applied to x)"*
$(\lambda x.x) \; x$	$((\lambda x.x) \; x)$	*"(lambda x to x) applied to x"*

λ-Calculus (cont'd)

β-Reduction

the term s (β-)reduces to the term t in one step, i.e.,

$$s \rightarrow_\beta t$$

iff there exist context C and terms u, v s.t.

$$s = C[(\lambda x.u)\ v] \qquad \text{and} \qquad t = C[u\{x/v\}]$$

λ-Calculus (cont'd)

β-Reduction

the term s (β-)reduces to the term t in one step, i.e.,

$$\overbrace{s \to_\beta t}^{(\beta\text{-})\text{step}}$$

iff there exist context C and terms u, v s.t.

$$s = C[(\lambda x.u)\ v] \qquad \text{and} \qquad t = C[u\{x/v\}]$$

λ-Calculus (cont'd)

β-Reduction

the term s (β-)reduces to the term t in one step, i.e.,

$$s \rightarrow_\beta t$$

iff there exist context C and terms u, v s.t.

$$s = C[(\lambda x.u)\ v] \qquad \text{and} \qquad t = C[u\{x/v\}]$$

Example

$$K \stackrel{\text{def}}{=} \lambda xy.x$$

$$I \stackrel{\text{def}}{=} \lambda x.x$$

$$\Omega \stackrel{\text{def}}{=} (\lambda x.x\ x)\ (\lambda x.x\ x)$$

Overview

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This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction,
reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing,

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, dependent types, monads

Overview

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Booleans and Conditionals

OCaml

- `true`
- `false`
- `if` b `then` t `else` e

Booleans and Conditionals

OCaml

- **`true`**
- **`false`**
- **`if`** b **`then`** t **`else`** e

λ-Calculus

Booleans and Conditionals

OCaml

- `true`
- `false`
- `if` b `then` t `else` e

λ-Calculus

- true $\stackrel{\text{def}}{=} \lambda xy.x$

Booleans and Conditionals

OCaml

- `true`
- `false`
- `if` b `then` t `else` e

λ-Calculus

- true $\stackrel{\text{def}}{=} \lambda xy.x$
- false $\stackrel{\text{def}}{=} \lambda xy.y$

Booleans and Conditionals

OCaml

- `true`
- `false`
- `if` b `then` t `else` e

λ-Calculus

- true $\stackrel{\text{def}}{=} \lambda xy.x$
- false $\stackrel{\text{def}}{=} \lambda xy.y$
- if $\stackrel{\text{def}}{=} \lambda xyz.x\ y\ z$

Booleans and Conditionals

OCaml

- `true`
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- `if` b `then` t `else` e

λ-Calculus

- true $\stackrel{\text{def}}{=} \lambda xy.x$
- false $\stackrel{\text{def}}{=} \lambda xy.y$
- if $\stackrel{\text{def}}{=} \lambda xyz.x\ y\ z$

Example

$$\text{if true } t\ e \rightarrow^{+}_{\beta}$$

$$\text{if false } t\ e \rightarrow^{+}_{\beta}$$

Booleans and Conditionals

OCaml

- **true**
- **false**
- **if** b **then** t **else** e

λ-Calculus

- $\text{true} \stackrel{\text{def}}{=} \lambda xy.x$
- $\text{false} \stackrel{\text{def}}{=} \lambda xy.y$
- $\text{if} \stackrel{\text{def}}{=} \lambda xyz.x\ y\ z$

Example

$$\text{if true } t\ e \to_\beta^+ \text{true } t\ e$$

$$\text{if false } t\ e \to_\beta^+$$

Booleans and Conditionals

OCaml

- `true`
- `false`
- `if` b `then` t `else` e

λ-Calculus

- true $\stackrel{\text{def}}{=} \lambda xy.x$
- false $\stackrel{\text{def}}{=} \lambda xy.y$
- if $\stackrel{\text{def}}{=} \lambda xyz.x\ y\ z$

Example

$$\text{if true } t\ e \rightarrow_\beta^+ \text{true } t\ e \rightarrow_\beta^+ t$$

$$\text{if false } t\ e \rightarrow_\beta^+$$

Booleans and Conditionals

OCaml

- `true`
- `false`
- **if** b **then** t **else** e

λ-Calculus

- $\text{true} \stackrel{\text{def}}{=} \lambda xy.x$
- $\text{false} \stackrel{\text{def}}{=} \lambda xy.y$
- $\text{if} \stackrel{\text{def}}{=} \lambda xyz.x\ y\ z$

Example

$$\text{if true } t\ e \rightarrow_\beta^+ \text{true } t\ e \rightarrow_\beta^+ t$$

$$\text{if false } t\ e \rightarrow_\beta^+ \text{false } t\ e$$

Booleans and Conditionals

OCaml

- `true`
- `false`
- **if** b **then** t **else** e

λ-Calculus

- true $\stackrel{\text{def}}{=} \lambda xy.x$
- false $\stackrel{\text{def}}{=} \lambda xy.y$
- if $\stackrel{\text{def}}{=} \lambda xyz.x\ y\ z$

Example

$$\text{if true } t\ e \rightarrow_\beta^+ \text{true } t\ e \rightarrow_\beta^+ t$$
$$\text{if false } t\ e \rightarrow_\beta^+ \text{false } t\ e \rightarrow_\beta^+ e$$

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

```
0
1
n
( + )
( * )
( ** )
```

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda f x.x$
`1`	
`n`	
`( + )`	
`( * )`	
`( ** )`	

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	
`( + )`	
`( * )`	
`( ** )`	

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	$\overline{\mathsf{n}} \stackrel{\text{def}}{=} \lambda fx.f^n\ x$
`( + )`	
`( * )`	
`( ** )`	

Natural Numbers

Definition

$s^0\ t \stackrel{\text{def}}{=} t \qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	$\overline{n} \stackrel{\text{def}}{=} \lambda fx.f^n\ x$
`( + )`	$\text{add} \stackrel{\text{def}}{=} \lambda mnfx.m\ f\ (n\ f\ x)$
`( * )`	
`( ** )`	

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	$\overline{n} \stackrel{\text{def}}{=} \lambda fx.f^n\ x$
`( + )`	$\text{add} \stackrel{\text{def}}{=} \lambda mnfx.m\ f\ (n\ f\ x)$
`( * )`	$\text{mul} \stackrel{\text{def}}{=} \lambda mnf.m\ (n\ f)$
`( ** )`	

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	$\overline{n} \stackrel{\text{def}}{=} \lambda fx.f^n\ x$
`( + )`	$\mathsf{add} \stackrel{\text{def}}{=} \lambda mnfx.m\ f\ (n\ f\ x)$
`( * )`	$\mathsf{mul} \stackrel{\text{def}}{=} \lambda mnf.m\ (n\ f)$
`( ** )`	$\mathsf{exp} \stackrel{\text{def}}{=} \lambda mn.n\ m$

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	$\overline{n} \stackrel{\text{def}}{=} \lambda fx.f^n\ x$
`( + )`	$\mathsf{add} \stackrel{\text{def}}{=} \lambda mnfx.m\ f\ (n\ f\ x)$
`( * )`	$\mathsf{mul} \stackrel{\text{def}}{=} \lambda mnf.m\ (n\ f)$
`( ** )`	$\mathsf{exp} \stackrel{\text{def}}{=} \lambda mn.n\ m$

Example

$$\mathsf{add}\ \overline{1}\ \overline{1} \to_\beta^*$$

Natural Numbers

Definition

$$s^0\ t \stackrel{\text{def}}{=} t \qquad\qquad s^{n+1}\ t \stackrel{\text{def}}{=} s\ (s^n\ t)$$

OCaml vs. λ-Calculus

`0`	$\overline{0} \stackrel{\text{def}}{=} \lambda fx.x$
`1`	$\overline{1} \stackrel{\text{def}}{=} \lambda fx.f\ x$
`n`	$\overline{\mathrm{n}} \stackrel{\text{def}}{=} \lambda fx.f^n\ x$
`( + )`	$\mathrm{add} \stackrel{\text{def}}{=} \lambda mnfx.m\ f\ (n\ f\ x)$
`( * )`	$\mathrm{mul} \stackrel{\text{def}}{=} \lambda mnf.m\ (n\ f)$
`( ** )`	$\mathrm{exp} \stackrel{\text{def}}{=} \lambda mn.n\ m$

Example

$$\mathrm{add}\ \overline{1}\ \overline{1} \to_\beta^* \overline{2}$$

Pairs

OCaml vs. λ-Calculus

```
fun x y -> (x,y)
fst
snd
```

Pairs

OCaml vs. λ-Calculus

`fun x y -> (x,y)` pair $\stackrel{\text{def}}{=} \lambda xyf.f\ x\ y$

`fst`

`snd`

Pairs

OCaml vs. λ-Calculus

`fun x y -> (x,y)`	pair $\stackrel{\text{def}}{=} \lambda xyf.f\ x\ y$
`fst`	fst $\stackrel{\text{def}}{=} \lambda p.p$ true
`snd`	

Pairs

OCaml vs. λ-Calculus

`fun x y -> (x,y)`	$\mathsf{pair} \stackrel{\text{def}}{=} \lambda xyf.f\ x\ y$
`fst`	$\mathsf{fst} \stackrel{\text{def}}{=} \lambda p.p\ \mathsf{true}$
`snd`	$\mathsf{snd} \stackrel{\text{def}}{=} \lambda p.p\ \mathsf{false}$

Pairs

OCaml vs. λ-Calculus

`fun x y -> (x,y)`	$\mathsf{pair} \stackrel{\mathsf{def}}{=} \lambda xyf.f\ x\ y$
`fst`	$\mathsf{fst} \stackrel{\mathsf{def}}{=} \lambda p.p\ \mathsf{true}$
`snd`	$\mathsf{snd} \stackrel{\mathsf{def}}{=} \lambda p.p\ \mathsf{false}$

Example

$$\mathsf{fst}\ (\mathsf{pair}\ \overline{\mathsf{m}}\ \overline{\mathsf{n}}) \to_\beta^*$$

Pairs

OCaml vs. λ-Calculus

`fun x y -> (x,y)`	pair $\stackrel{\text{def}}{=} \lambda xyf.f\ x\ y$
`fst`	fst $\stackrel{\text{def}}{=} \lambda p.p$ true
`snd`	snd $\stackrel{\text{def}}{=} \lambda p.p$ false

Example

$$\mathsf{fst}\ (\mathsf{pair}\ \overline{\mathsf{m}}\ \overline{\mathsf{n}}) \to_\beta^* \overline{\mathsf{m}}$$

Lists

OCaml vs. λ-Calculus

```
::
hd
tl
[]
fun x -> x = []
```

Lists

OCaml vs. λ-Calculus

`::`	$\mathsf{cons} \stackrel{\mathsf{def}}{=} \lambda xy.$	$\mathsf{pair}\ x\ y$
`hd`		
`tl`		
`[]`		
`fun x -> x = []`		

Lists

OCaml vs. λ-Calculus

`::`	$\text{cons} \stackrel{\text{def}}{=} \lambda xy.\text{pair false (pair } x\ y)$
`hd`	
`tl`	
`[]`	
`fun x -> x = []`	

Lists

OCaml vs. λ-Calculus

`::`	$\mathsf{cons} \stackrel{\text{def}}{=} \lambda xy.\mathsf{pair}\ \mathsf{false}\ (\mathsf{pair}\ x\ y)$
`hd`	$\mathsf{hd} \stackrel{\text{def}}{=} \lambda z.\mathsf{fst}\ (\mathsf{snd}\ z)$
`tl`	
`[]`	
`fun x -> x = []`	

Lists

OCaml vs. λ-Calculus

OCaml	λ-Calculus
`::`	$\mathsf{cons} \stackrel{\text{def}}{=} \lambda xy.\mathsf{pair}\ \mathsf{false}\ (\mathsf{pair}\ x\ y)$
`hd`	$\mathsf{hd} \stackrel{\text{def}}{=} \lambda z.\mathsf{fst}\ (\mathsf{snd}\ z)$
`tl`	$\mathsf{tl} \stackrel{\text{def}}{=} \lambda z.\mathsf{snd}\ (\mathsf{snd}\ z)$
`[]`	
`fun x -> x = []`	

Lists

OCaml vs. λ-Calculus

`::`	$\mathsf{cons} \stackrel{\text{def}}{=} \lambda xy.\mathsf{pair}\ \mathsf{false}\ (\mathsf{pair}\ x\ y)$
`hd`	$\mathsf{hd} \stackrel{\text{def}}{=} \lambda z.\mathsf{fst}\ (\mathsf{snd}\ z)$
`tl`	$\mathsf{tl} \stackrel{\text{def}}{=} \lambda z.\mathsf{snd}\ (\mathsf{snd}\ z)$
`[]`	$\mathsf{nil} \stackrel{\text{def}}{=} \lambda x.x$
`fun x -> x = []`	

Lists

OCaml vs. λ-Calculus

`::`	$\text{cons} \stackrel{\text{def}}{=} \lambda xy.\text{pair false } (\text{pair } x\ y)$
`hd`	$\text{hd} \stackrel{\text{def}}{=} \lambda z.\text{fst } (\text{snd } z)$
`tl`	$\text{tl} \stackrel{\text{def}}{=} \lambda z.\text{snd } (\text{snd } z)$
`[]`	$\text{nil} \stackrel{\text{def}}{=} \lambda x.x$
`fun x -> x = []`	$\text{null} \stackrel{\text{def}}{=} \text{fst}$

Lists

OCaml vs. λ-Calculus

`::`	$\mathsf{cons} \stackrel{\text{def}}{=} \lambda xy.\mathsf{pair}\ \mathsf{false}\ (\mathsf{pair}\ x\ y)$
`hd`	$\mathsf{hd} \stackrel{\text{def}}{=} \lambda z.\mathsf{fst}\ (\mathsf{snd}\ z)$
`tl`	$\mathsf{tl} \stackrel{\text{def}}{=} \lambda z.\mathsf{snd}\ (\mathsf{snd}\ z)$
`[]`	$\mathsf{nil} \stackrel{\text{def}}{=} \lambda x.x$
`fun x -> x = []`	$\mathsf{null} \stackrel{\text{def}}{=} \mathsf{fst}$

Example

$$\mathsf{null}\ \mathsf{nil} \to_\beta^*$$

Lists

OCaml vs. λ-Calculus

OCaml	λ-Calculus
`::`	$\mathsf{cons} \stackrel{\text{def}}{=} \lambda xy.\mathsf{pair}\ \mathsf{false}\ (\mathsf{pair}\ x\ y)$
`hd`	$\mathsf{hd} \stackrel{\text{def}}{=} \lambda z.\mathsf{fst}\ (\mathsf{snd}\ z)$
`tl`	$\mathsf{tl} \stackrel{\text{def}}{=} \lambda z.\mathsf{snd}\ (\mathsf{snd}\ z)$
`[]`	$\mathsf{nil} \stackrel{\text{def}}{=} \lambda x.x$
`fun x -> x = []`	$\mathsf{null} \stackrel{\text{def}}{=} \mathsf{fst}$

Example

$$\mathsf{null}\ \mathsf{nil} \to_\beta^* \mathsf{true}$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                                else 1 + length(tl x)
```

λ-Calculus

$$\text{length} \stackrel{\text{def}}{=} \lambda x.\text{if } (\text{null } x)\ \overline{0}\ (\text{add } \overline{1}\ (\text{length } (\text{tl } x)))$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                              else 1 + length(tl x)
```

λ-Calculus

$$\text{length} \stackrel{\text{def}}{=} \lambda x.\text{if}\ (\text{null}\ x)\ \overline{0}\ (\text{add}\ \overline{1}\ (\text{length}\ (\text{tl}\ x)))$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                              else 1 + length(tl x)
```

λ-Calculus

$$\text{length} \stackrel{\text{def}}{=} \lambda f x.\text{if } (\text{null } x)\ \overline{0}\ (\text{add } \overline{1}\ (f\ (\text{tl } x)))$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                             else 1 + length(tl x)
```

λ-Calculus

$$\text{length} \stackrel{\text{def}}{=} \mathsf{Y}\ (\lambda f x.\text{if}\ (\text{null}\ x)\ \overline{0}\ (\text{add}\ \overline{1}\ (f\ (\text{tl}\ x))))$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                                   else 1 + length(tl x)
```

λ-Calculus

$$\text{length} \stackrel{\text{def}}{=} \mathsf{Y}\ (\lambda f x.\text{if}\ (\text{null}\ x)\ \overline{0}\ (\text{add}\ \overline{1}\ (f\ (\text{tl}\ x))))$$

Definition (Y-combinator)

$$\mathsf{Y} \stackrel{\text{def}}{=} \lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))$$

Y has fixed point property, i.e., for all $t \in \mathcal{T}(\mathcal{V})$

$$\mathsf{Y}\ t \leftrightarrow^* t\ (\mathsf{Y}\ t)$$

Recursion

OCaml

```
let rec length x = if x = [] then 0
                               else 1 + length(tl x)
```

λ-Calculus

$$\text{length} \stackrel{\text{def}}{=} \text{Y}\ (\lambda f x.\text{if}\ (\text{null}\ x)\ \overline{0}\ (\text{add}\ \overline{1}\ (f\ (\text{tl}\ x))))$$

Definition (Y-combinator)

$$\text{Y} \stackrel{\text{def}}{=} \lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))$$

Y has fixed point property, i.e., for all $t \in \mathcal{T}(\mathcal{V})$

$$\text{Y}\ t \leftrightarrow^* t\ (\text{Y}\ t)$$

Overview

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Example

- consider `let d x = x + x`
- the term `d (d 2)` can be evaluated as follows

```
d (d 2)
```

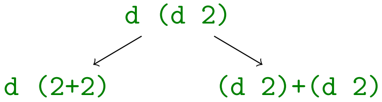
Example

- consider `let d x = x + x`
- the term `d (d 2)` can be evaluated as follows

```
        d (d 2)
       ↙
d (2+2)
```

Example

- consider `let d x = x + x`
- the term `d (d 2)` can be evaluated as follows



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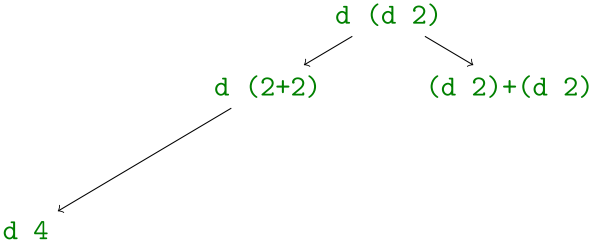

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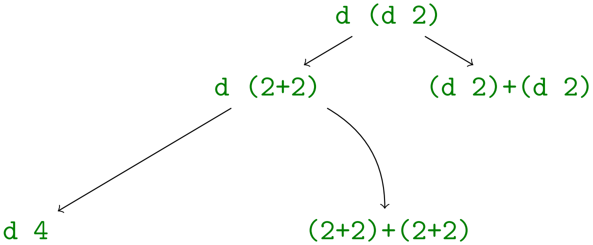

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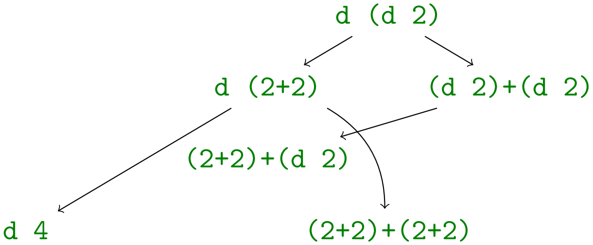

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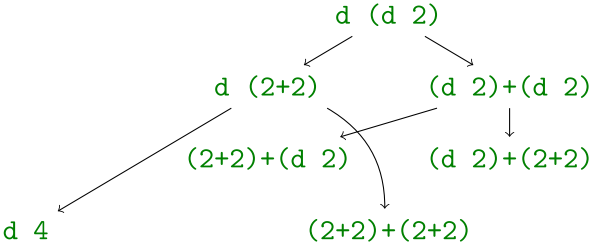

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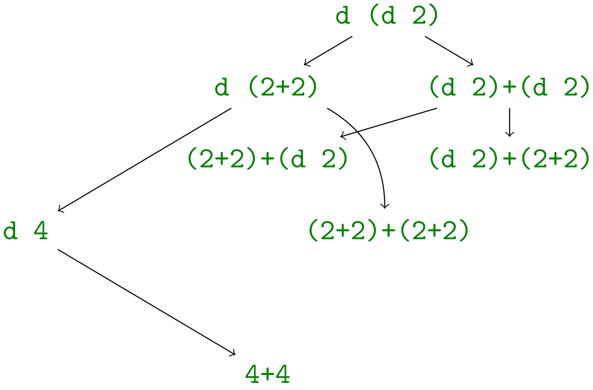

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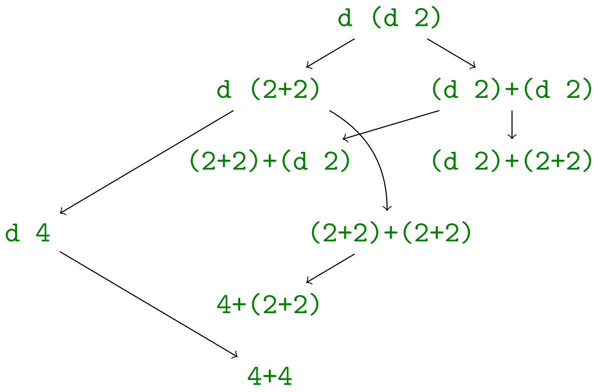

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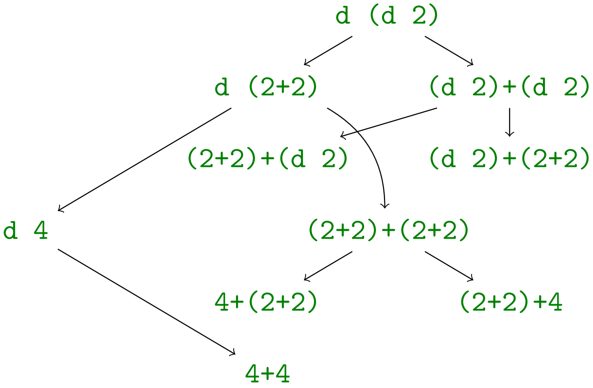

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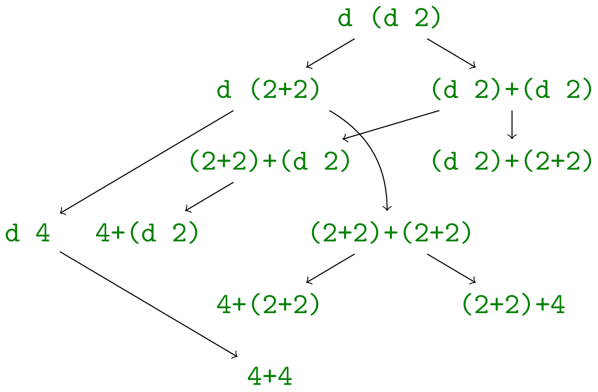

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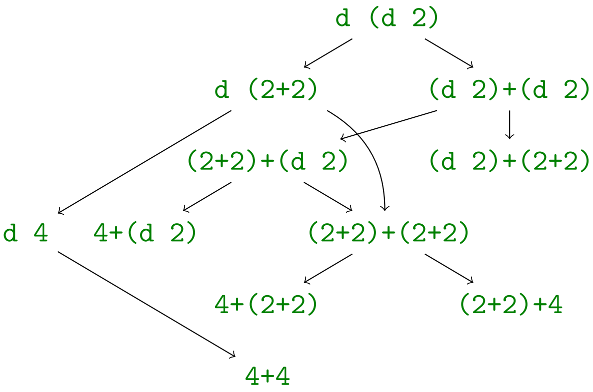

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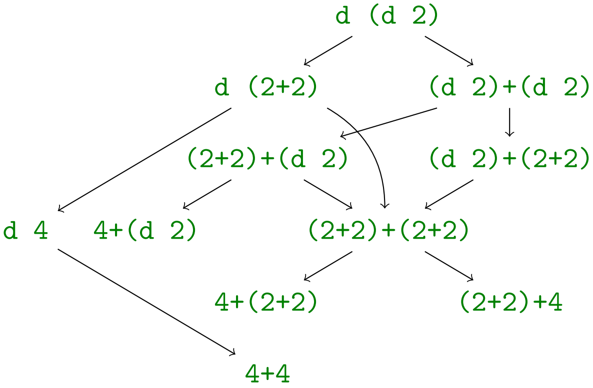

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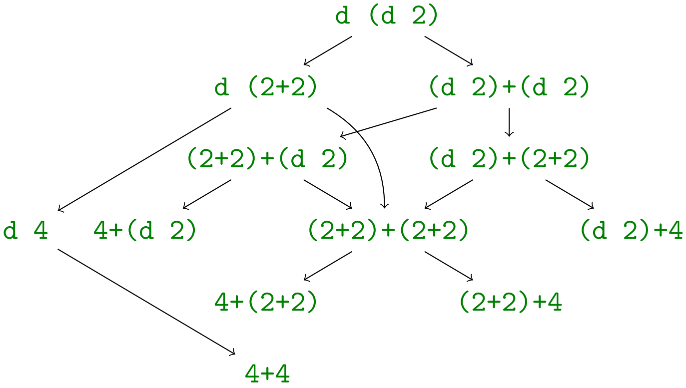

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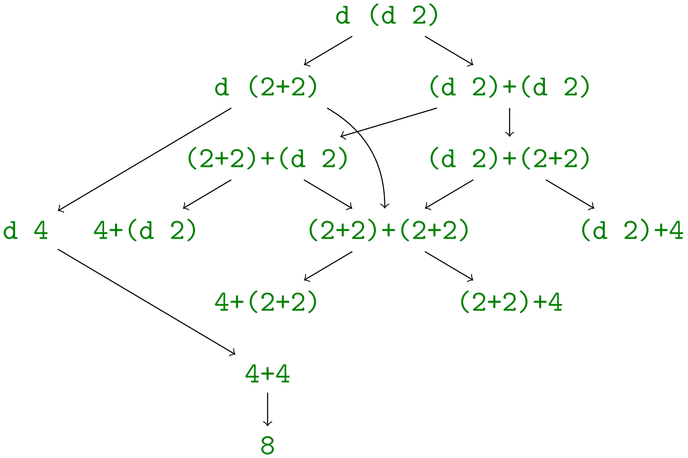

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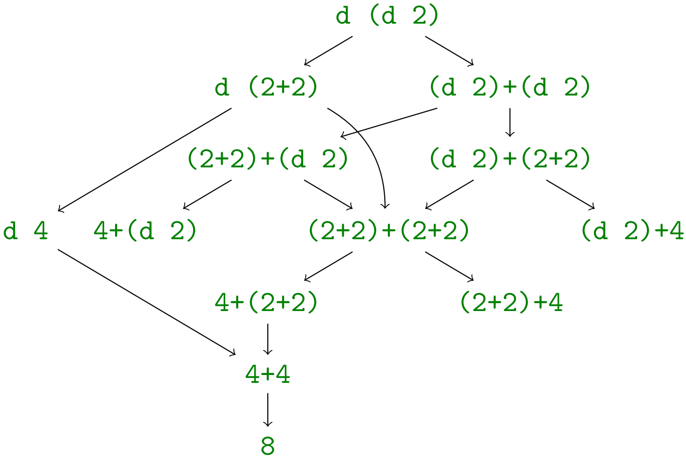

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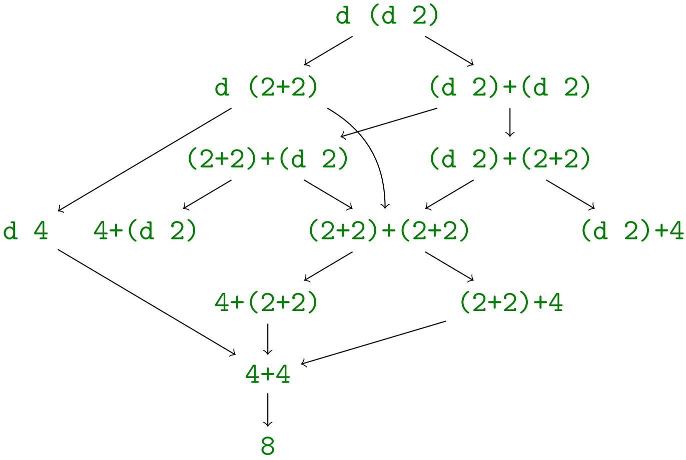

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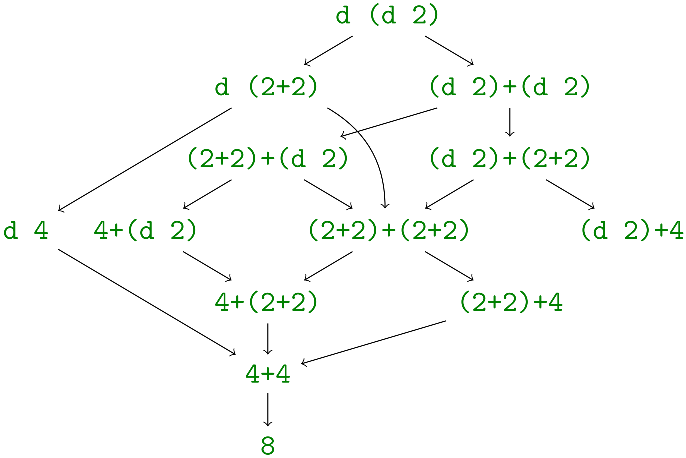

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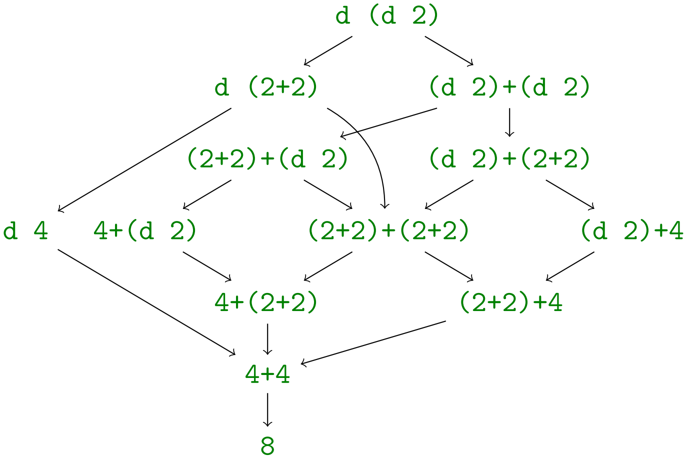

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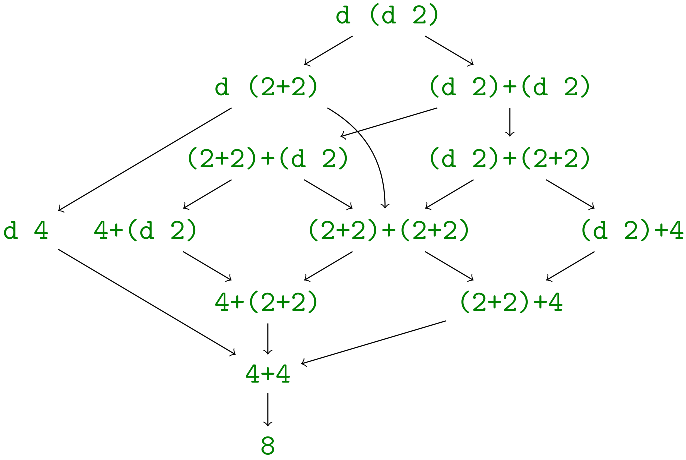

Strategies

Strategy

- fixes evaluation order
- examples: call-by-value and call-by-name

Example

```
let d x = x + x
```

- call-by-value:

```
d (d 2) → d (2+2)
        → d 4
        → 4 + 4
        → 8
```

- call-by-name:

```
d (d 2) → (d 2)+(d 2)
        → (2+2)+(d 2)
        → 4+(d 2)
        → 4+(2+2)
        → 4+4
        → 8
```

(Leftmost) Innermost Reduction

- always reduce (leftmost) innermost redex

Definition

redex t of term u is innermost if it does not contain a redex as proper subterm, i.e.,

$$\nexists s \in \mathcal{S}\text{ub}(t) \text{ s.t. } s \neq t \text{ and } s \text{ is a redex}$$

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- use innermost reduction
- corresponds to strict (or eager) evaluation, e.g., OCaml
- slight modification: only reduce terms that are not in WHNF

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Call-by-Name

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