Functional Programming

WS 2015/16

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week 7

Overview

- Week 7 - Induction
  - Summary of Week 6
  - Mathematical Induction
  - Induction Over Lists
  - Structural Induction

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Rewrite Strategies

Outermost

- choose the (leftmost) outermost redex
- redex is outermost if not subterm of different redex

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Call-by-name

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WHNF (Intuition)

Thou shalt not reduce below lambda.

Evaluation Strategies

Lazy

- call-by-name + sharing
- only evaluate if necessary
- e.g. Haskell

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Lazy

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Strict/Eager

- call-by-value
- evaluate arguments before calling a function
- e.g. OCaml (also support for laziness)

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This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction, reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing,

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, dependent types, monads

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When?

Goal

"*prove that some property P holds for all natural numbers*"

Formally

$$\forall n.P(n) \qquad (\text{where } n \in \mathbb{N})$$

How?

2 goals to show

1. $P(0)$
2. $\forall k.(P(k) \rightarrow P(k+1))$

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1. $P(0)$
2. $\forall k.(P(k) \rightarrow P(k+1))$

Gives

$$(P(0) \wedge \forall k.(P(k) \rightarrow P(k+1))) \rightarrow \forall n.P(n)$$

Why Does This Work?

We have

- $P(0)$
- $\forall k.(P(k) \rightarrow P(k+1))$

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- have $P(n-1) \rightarrow P(n)$
- hence $P(n)$

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anything that depends on some variable and is either true or false can be seen as function `p : 'a -> bool`

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  show: $P(k+1)$

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$$1 + 2 + \cdots + (k+1)$$

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$$
\begin{aligned}
1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\
&\overset{\text{IH}}{=} \frac{k \cdot (k+1)}{2} + (k+1) \\
&= \frac{(k+1) \cdot (k+2)}{2}
\end{aligned}
$$

Remark

- of course the base case can be changed
- e.g., if base case $P(1)$, property holds for all $n \geq 1$

Overview

Recall

Type

```
type 'a list = [] | (::) of 'a * 'a list
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Note

- lists are recursive structures
- base case: `[]`
- step case: x `::` xs

Induction Principle on Lists

Intuition

- to show $P(xs)$ for all lists xs
- show base case: $P(\texttt{[]})$
- show step case: $P(xs) \rightarrow P(x \texttt{ : : } xs)$ for arbitrary x and xs

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Formally

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Remarks

- $y \texttt{ : } \beta$ reads 'y *is of type* β'
- for lists, P can be seen as function `p : 'a list -> bool`

Example - `Lst.append`

Recall

```
let rec (@) xs ys = match xs with
  | []    -> ys
  | x::xs -> x :: (xs @ ys)
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Proof.

Blackboard □

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length of combined list equals sum of lengths, i.e.,

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          | App of (term * term)
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Induction Principle

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Induction Principle

- for every non-recursive constructor there is a base case

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  - base case: `Var x`

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  - step case: `Abs`(x, t)

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- for every non-recursive constructor there is a base case
  - base case: `Var x`
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  - step case: `Abs`(x, t)
  - step case: `App`(s, t)

Induction Principle on General Structures

Intuition

- to show $P(s)$ for all structures s
- show base cases
- show step cases

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```
type 'a btree = Empty | Node of ('a btree * 'a * 'a btree)
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Induction Principle

$$\begin{array}{c} (P(\texttt{Empty}) \wedge \\ \forall v \quad .\forall l \quad .\forall r \quad . \\ \\ ((P(l) \wedge P(r)) \rightarrow P(\texttt{Node}(l, v, r)))) \\ \rightarrow \\ \forall t \quad .P(t) \end{array}$$

Recall

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Induction Principle

$$(P(\texttt{Empty}) \wedge$$
$$\forall v : \alpha.\forall l : \alpha\ \texttt{btree}.\forall r : \alpha\ \texttt{btree}.$$

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$$\rightarrow$$
$$\forall t : \alpha\ \texttt{btree}.P(t)$$

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$$
\begin{gathered}
(P(\texttt{Empty}) \wedge \\
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\overbrace{((P(l) \wedge P(r))}^{\text{IH}} \rightarrow P(\texttt{Node}(l, v, r)))) \\
\rightarrow \\
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\end{gathered}
$$

Example - Trees

Definition (Perfect Binary Trees)

binary tree is perfect if all leafs have same depth

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Recall

```
let rec perfect = function
  | Empty -> true
  | Node(l,_,r) -> height l = height r && perfect l && perfect r

let rec height = function
  | Empty        -> 0
  | Node(l,_,r) -> max (height l) (height r) + 1

let rec size = function
  | Empty        -> 0
  | Node(l,_,r) -> size l + size r + 1
```

Example - Trees (cont'd)

Lemma

perfect binary tree t of height n has exactly $2^n - 1$ nodes

Example - Trees (cont'd)

Lemma

perfect binary tree t of height n has exactly $2^n - 1$ nodes

Proof.

To show: $P(t) = (\texttt{perfect}\ t \rightarrow (\texttt{size}\ t = 2^{(\texttt{height}\ t)} - 1))$

Blackboard