Functional Programming Exercises Week 6

(for November 18, 2016)

1. Read Chapters 5.3 - 5.6 of the lecture notes.

2. Exercise 5.11

We use the following basic facts:

Lemma 1 (composition of contexts)**.** *For all contexts C, D, there exists a context E such that for all terms t, $C[D[t]] = E[t]$.*

Proof. By induction on C. □

Corollary 1. *If $s \to_\beta t$ then $C[s] \to_\beta C[t]$ for all contexts C. If $s \to_\beta^* t$ then $C[s] \to_\beta^* C[t]$ for all contexts C.*

Proof. The first statement follows by definition of $\to_\beta$ and Lemma 1. The second statement follows by induction on the sequence $s \to_\beta^* t$. □

The exercise is proved as follows:

Base Case ($m = 0$). Using Definition 5.2 we get

$$(\lambda x. f^n\ x)^0\ x = x \to_\beta^* x$$

Step Case. The IH is

$$(\lambda x. f^n\ x)^m\ x \to_\beta^* f^{nm}\ x$$

The goal is to prove

$$(\lambda x. f^n\ x)^{m+1}\ x \to_\beta^* f^{n(m+1)}\ x$$

By Definition 5.2, we have

$$(\lambda x. f^n\ x)^{m+1}\ x = (\lambda x. f^n\ x)\ (\underline{(\lambda x. f^n\ x)^m\ x})$$

We want to apply the IH on the underlined λ-term. This is possible by Corollary 1.

Finally we apply the IH and get

$$\begin{aligned}
(\lambda x. f^n\ x)\ (\underline{(\lambda x. f^n\ x)^m\ x}) &\to_\beta^* (\lambda x. f^n\ x)\ (\underline{f^{nm}\ x}) && \text{(IH)}\\
&\to_\beta f^n\ (f^{nm}\ x) && (\beta\text{-step})\\
&= f^{n+nm}\ x && \text{(Exercise 5.10)}\\
&= f^{n(m+1)}\ x && \text{(arithmetic)}
\end{aligned}$$

3. Exercise 5.14

Remember that every λ-term that is not an application is in WHNF.

a) $\mathsf{add}\ \overline{2}\ \overline{3}$

$$
\begin{aligned}
&= (\lambda mnfx.m\ f\ (n\ f\ x))\ (\lambda fx.f\ (f\ x))\ (\lambda fx.f\ (f\ (f\ x)))\\
&\to_\beta (\lambda nfx.(\lambda fx.f\ (f\ x))\ f\ (n\ f\ x))\ (\lambda fx.f\ (f\ (f\ x)))\\
&\to_\beta (\lambda nfx.(\lambda x.f\ (f\ x))\ (n\ f\ x))\ (\lambda fx.f\ (f\ (f\ x)))\\
&\to_\beta (\lambda nfx.f\ (f\ (n\ f\ x)))\ (\lambda fx.f\ (f\ (f\ x)))\\
&\to_\beta \lambda fx.f\ (f\ ((\lambda fx.f\ (f\ (f\ x)))\ f\ x))
\end{aligned}
$$

b) $\mathsf{add}\ \overline{2}\ \overline{3}$

$$
\begin{aligned}
&= (\lambda mnfx.m\ f\ (n\ f\ x))\ (\lambda fx.f\ (f\ x))\ (\lambda fx.f\ (f\ (f\ x)))\\
&\to_\beta (\lambda nfx.(\lambda fx.f\ (f\ x))\ f\ (n\ f\ x))\ (\lambda fx.f\ (f\ (f\ x)))\\
&\to_\beta \lambda fx.(\lambda fx.f\ (f\ x))\ f\ ((\lambda fx.f\ (f\ (f\ x)))\ f\ x)
\end{aligned}
$$

4. Exercise 5.16 The subterms of Y are:

$$
\begin{aligned}
t_1 &= \lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)),\\
t_2 &= (\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x)),\\
t_3 &= \lambda x.f\ (x\ x),\\
t_4 &= f\ (x\ x),\\
t_5 &= f,\\
t_6 &= x\ x,\\
t_7 &= x.
\end{aligned}
$$

All but t_1 are proper subterms of Y. The sets of variables occurring in, free variables of, and bound variables of t_i are as follows:

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\mathcal{V}\mathsf{ar}(\cdot)$	$\{f, x\}$	$\{f, x\}$	$\{f, x\}$	$\{f, x\}$	$\{f\}$	$\{x\}$	$\{x\}$
$\mathcal{FV}\mathsf{ar}(\cdot)$	$\varnothing$	$\{f\}$	$\{f\}$	$\{f, x\}$	$\{f\}$	$\{x\}$	$\{x\}$
$\mathcal{BV}\mathsf{ar}(\cdot)$	$\{f, x\}$	$\{x\}$	$\{x\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

5. Exercise 5.21