Functional Programming
Exercises Week 8
(for December 2, 2016)

1. —

2. —

3.

$$(\forall a.P(\texttt{Leaf}\ a)) \land (\forall lr.P(l) \land P(r) \to P(\texttt{Node}\ (l,r))) \to P(t)$$

or with explicit types:

$$(\forall a{:}{:}\texttt{'a}.P(\texttt{Leaf}\ a)) \land (\forall lr{:}{:}\texttt{'a ltree}.P(l) \land P(r) \to P(\texttt{Node}\ (l,r))) \to \forall t{:}{:}\texttt{'a ltree}.P(t)$$

4.
```
let rec flatten_ltree = function
  | Leaf a -> [a]
  | Node (l, r) -> flatten_ltree l @ flatten_ltree r
```

5.
```
let rec join_ltree = function
  | Leaf a -> a
  | Node (l, r) -> Node (join_ltree l, join_ltree r)
```

6. ($\star$) We write `fl` for `flatten_ltree` and `jl` for `join_ltree`. We need to show that

$$\texttt{fl}\ (\texttt{jl}\ t) = \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ t))$$

The proof is by structural induction on t:

$$P(t):\ \texttt{fl}\ (\texttt{jl}\ t) = \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ t))$$

`Leaf` case: (base case) We show the property $P(\texttt{Leaf}\ a)$:

$$\begin{aligned}\texttt{fl}\ (\texttt{jl}\ (\texttt{Leaf}\ a)) = \texttt{fl}\ a = \texttt{concat}\ [\texttt{fl}\ a] = \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ [a])\\ = \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ (\texttt{Leaf}\ a))\end{aligned}$$

`Node` (l,r) case: (step case) To prove $P(l) \land P(r) \to P(\texttt{Node}\ (l,r))$, assume the two induction hypotheses:

$$\texttt{fl}\ (\texttt{jl}\ l) = \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ l)) \qquad (\text{IH}_l)$$
$$\texttt{fl}\ (\texttt{jl}\ r) = \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ r)) \qquad (\text{IH}_r)$$

We show the property $P(\texttt{Node}\ (l, r))$:

$$\begin{aligned}
\texttt{fl}\ (\texttt{jl}\ (\texttt{Node}\ (l, r))) &= \texttt{fl}\ (\texttt{Node}\ (\texttt{jl}\ l, \texttt{jl}\ r)) = \texttt{fl}\ (\texttt{jl}\ l)\ \texttt{@}\ \texttt{fl}\ (\texttt{jl}\ r) \\
&= \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ l))\ \texttt{@}\ \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ r)) \\
&= \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ l)\ \texttt{@}\ \texttt{map}\ \texttt{fl}\ (\texttt{fl}\ r)) \\
&= \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ l\ \texttt{@}\ \texttt{fl}\ r)) \\
&= \texttt{concat}\ (\texttt{map}\ \texttt{fl}\ (\texttt{fl}\ (\texttt{Node}\ (l, r))))
\end{aligned}$$

In addition to (IH_l) and (IH_r) we used Exercise 2 and the following useful property of `map` that can be proved by an easy induction on xs:

$$\texttt{map}\ f\ (xs\ \texttt{@}\ ys) = \texttt{map}\ f\ xs\ \texttt{@}\ \texttt{map}\ f\ ys$$