Functional Programming

WS 2016/17

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week 7

Overview

- Week 7 - Induction
  - Summary of Week 6
  - Mathematical Induction
  - Induction Over Lists
  - Structural Induction

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Rewrite Strategies

Outermost

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WHNF (Intuition)

Thou shalt not reduce below lambda.

Evaluation Strategies

Lazy

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- only evaluate if necessary
- e.g. Haskell

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Strict/Eager

- call-by-value
- evaluate arguments before calling a function
- e.g. OCaml (also support for lazyness)

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This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction, reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing,

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, dependent types, monads

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When?

Goal

"*prove that some property P holds for all natural numbers*"

Formally

$$\forall n.P(n) \qquad (\text{where } n \in \mathbb{N})$$

How?

2 goals to show

1. $P(0)$
2. $\forall k.(P(k) \rightarrow P(k+1))$

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1. $P(0)$
2. $\forall k.(P(k) \rightarrow P(k+1))$

Gives

$$(P(0) \wedge \forall k.(P(k) \rightarrow P(k+1))) \rightarrow \forall n.P(n)$$

Why Does This Work?

We have

- $P(0)$
- $\forall k.(P(k) \rightarrow P(k+1))$

Why Does This Work?

We have

- $P(0)$ "property P holds for 0"
- $\forall k.(P(k) \to P(k+1))$

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- have $P(0) \rightarrow P(1)$

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- have $P(1) \rightarrow P(2)$

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- have $P(n-1)$
- have $P(n-1) \to P(n)$
- hence $P(n)$

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anything that depends on some variable and is either true or false can be seen as function `p : 'a -> bool`

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  IH: $P(k) = (1 + 2 + \cdots + k = \frac{k \cdot (k+1)}{2})$
  show: $P(k+1)$

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$$1 + 2 + \cdots + (k+1)$$

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$$1 + 2 + \cdots + (k+1) = (1 + 2 + \cdots + k) + (k+1)$$

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$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &\overset{\text{IH}}{=} \frac{k\cdot(k+1)}{2} + (k+1) \end{aligned}$$

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\end{aligned}
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Remark

- of course the base case can be changed
- e.g., if base case $P(1)$, property holds for all $n \geq 1$

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Recall

Type

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type 'a list = [] | (::) of 'a * 'a list
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Note

- lists are recursive structures
- base case: `[]`
- step case: x `::` xs

Induction Principle on Lists

Intuition

- to show $P(xs)$ for all lists xs
- show base case: $P(\texttt{[]})$
- show step case: $P(xs) \to P(x :: xs)$ for arbitrary x and xs

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Formally

$$(P(\texttt{[]}) \wedge \forall x \;.\forall xs \;.(P(xs) \rightarrow P(x :: xs))) \rightarrow \forall ls \;.P(ls)$$

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Remarks

- $y : \beta$ reads '*y is of type β*'
- for lists, P can be seen as function `p : 'a list -> bool`

Example - Lst.append

Recall

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let rec (@) xs ys = match xs with
 | []    -> ys
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Proof.

Blackboard □

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sum of lengths equals length of combined list, i.e.,

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          | Abs of (var  * term)
          | App of (term * term)
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Induction Principle

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  - base case: `Var x`

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  - base case: `Var x`
- for every recursive constructor there is a step case
  - step case: `Abs`(x, t)
  - step case: `App`(s, t)

Induction Principle on General Structures

Intuition

- to show $P(s)$ for all structures s
- show base cases
- show step cases

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```
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Induction Principle

$$
\begin{gathered}
(P(\texttt{Empty}) \land \\
\forall v \quad . \forall l \qquad . \forall r \qquad . \\
((P(l) \land P(r)) \to P(\texttt{Node}(l, v, r)))) \\
\to \\
\forall t \qquad . P(t)
\end{gathered}
$$

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$$(P(\texttt{Empty}) \wedge$$
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Example - Trees

Definition (Perfect Binary Trees)

binary tree is perfect if all leaf nodes have same height

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Recall

```
let rec perfect = function
  | Empty       -> true
  | Node(l,_,r) -> height l = height r && perfect l && perfect r

let rec height = function
  | Empty       -> 0
  | Node(l,_,r) -> max (height l) (height r) + 1

let rec size = function
  | Empty       -> 0
  | Node(l,_,r) -> size l + size r + 1
```

Example - Trees (cont'd)

Lemma

perfect binary tree t of height n has exactly $2^n - 1$ nodes

Example - Trees (cont'd)

Lemma

perfect binary tree t of height n has exactly $2^n - 1$ *nodes*

Proof.

To show: $P(t) = \big(\texttt{perfect}\ t \rightarrow (\texttt{size}\ t = 2^{(\texttt{height}\ t)} - 1)\big)$

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