Functional Programming

WS 2016/17

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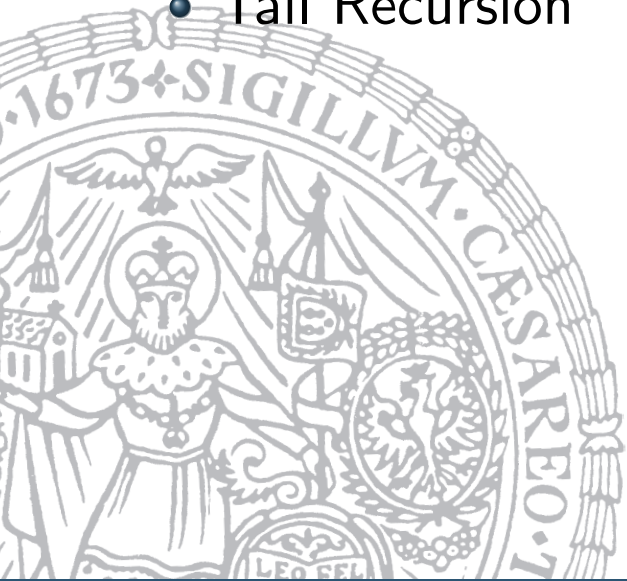
week 9

Overview

- Week 9 - Efficiency
  - Summary of Week 8
  - Fibonacci Numbers
  - Tupling
  - Tail Recursion



Overview

Isabelle/HOL

- Proof Assistant
- small trusted kernel
  - LCF approach
- Higher-order logic
- Proofs by induction, simplifier and automation (tableaux)
- Code generation
- Growing number of applications
  - Compilers, OS, Hardware, Cryptography
  - Actual mathematical proofs

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This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction,
reasoning about functional programs

Practice II

efficiency, tail-recursion, combinator-parsing,

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, dependent types, monads

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Mathematical

Definition (n-th Fibonacci number)

$$\text{fib}(n) \stackrel{\text{def}}{=} \begin{cases} n & \text{if } n \leq 1 \\ \text{fib}(n-1) + \text{fib}(n-2) & \text{otherwise} \end{cases}$$

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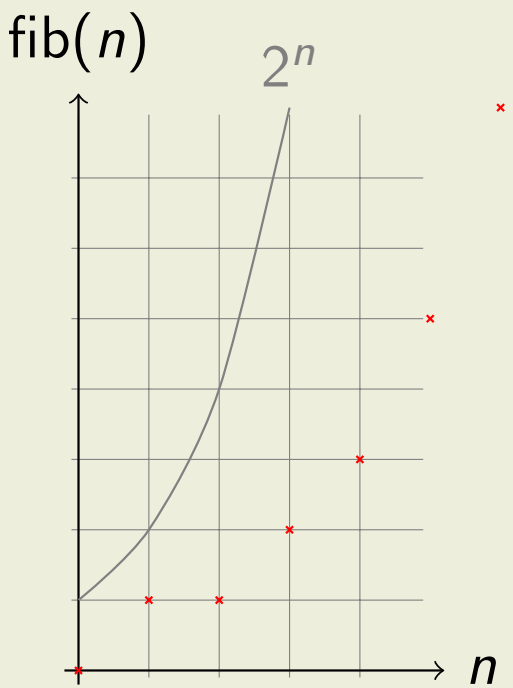
Graph



Mathematical (cont'd)

Example

0, 1

Mathematical (cont'd)

Example

0, 1, 1

Mathematical (cont'd)

Example

0, 1, 1, 2

Mathematical (cont'd)

Example

0, 1, 1, 2, 3

Mathematical (cont'd)

Example

0, 1, 1, 2, 3, 5

Mathematical (cont'd)

Example

0, 1, 1, 2, 3, 5, 8

Mathematical (cont'd)

Example

0, 1, 1, 2, 3, 5, 8, 13

Mathematical (cont'd)

Example

0, 1, 1, 2, 3, 5, 8, 13, 21

Mathematical (cont'd)

Example

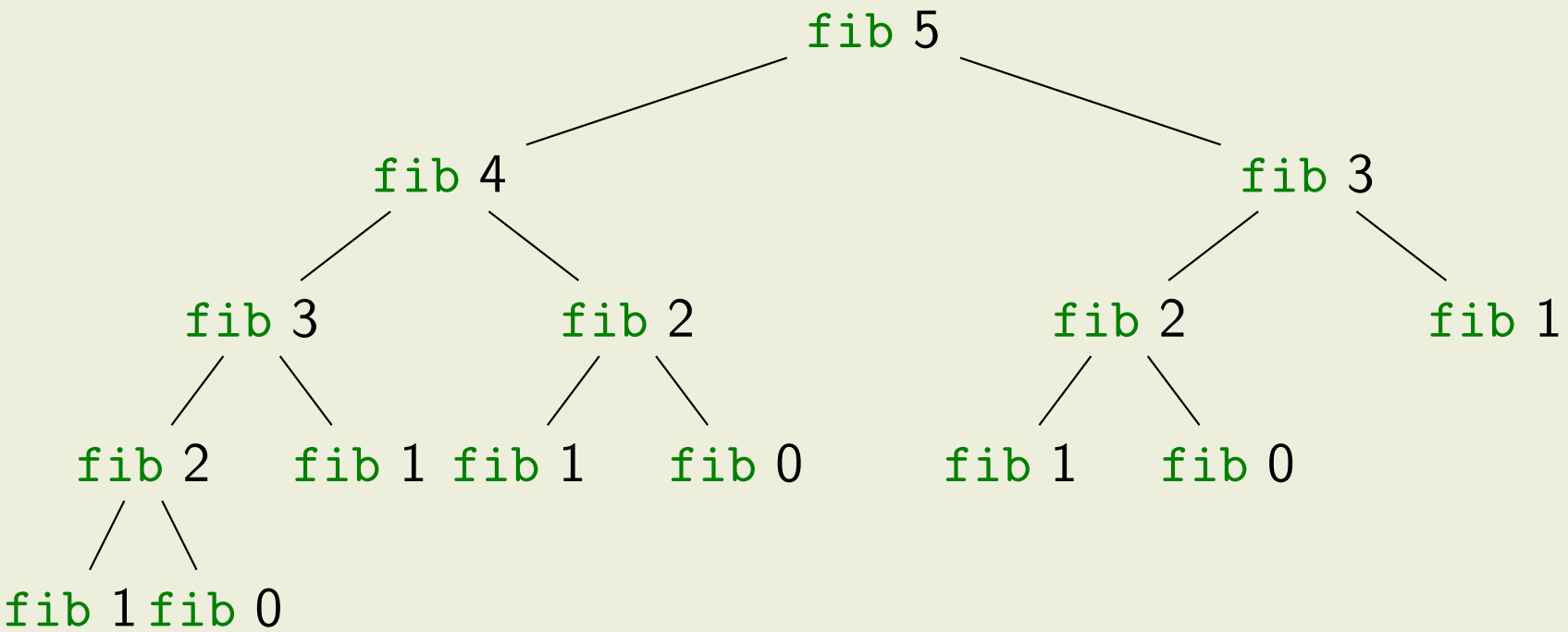
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181 ,6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155, 165580141, 267914296, 433494437, 701408733, 1134903170, 1836311903, 2971215073, . . .

OCaml

Definition

```
let rec fib n = if n < 2 then n else fib(n-1) + fib(n-2)
```

Example



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Tupling

Idea

- use tuples to return more than one result
- make results available as return values instead of recomputing them

Fibonacci Numbers

Example

```
let rec fibpair n = if n < 1 then (0, 1) else
  let (f1, f2) = fibpair (n - 1) in (f2, f1 + f2);;
```

Fibonacci Numbers

Example

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let rec fibpair n = if n < 1 then (0, 1) else
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Lemma

$$\texttt{fibpair}(n) = (\texttt{fib}\ n, \texttt{fib}(n+1))$$

Fibonacci Numbers

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Lemma

$$\texttt{fibpair}(n) = (\texttt{fib}\ n, \texttt{fib}(n+1))$$

Proof.

Blackboard □

A Second Example

Goal

compute average value of an integer list (module IntLst)

Naive Approach

- `let average xs = sum xs / Lst.length xs`

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- 2 traversals of `xs` are done

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Combined Function

- ```
  let rec sumlen = function
    | []    -> (0,0)
    | x::xs -> let (sum,len) = sumlen xs in (sum+x,len+1)
  ```
- `let average1 xs = let (sum,len) = sumlen xs in sum/len`
  ```

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compute average value of an integer list (module IntLst)

Naive Approach

- `let average xs = sum xs / Lst.length xs`
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Combined Function

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    | x::xs -> let (sum,len) = sumlen xs in (sum+x,len+1)
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- `let average1 xs = let (sum,len) = sumlen xs in sum/len`
- one traversal of `xs` suffices
  ```

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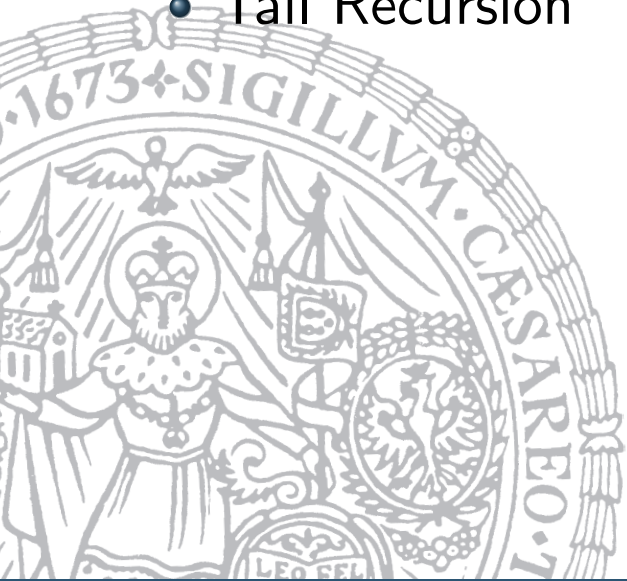

Recursion vs. Tail Recursion

Idea

- a function calling itself is recursive
- functions that mutually call each other are mutually recursive

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Definition (Tail recursion)

a function is called tail recursive if the recursive call is last action in the function body

Examples

Length

- ```
  let rec length = function []    -> 0
                          | _::xs -> 1 + length xs
  ```
  ```

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Even/Odd

- ```
  let rec is_even = function 0 -> true
                           | n -> is_odd (n - 1)
  and is_odd      = function 0 -> false
                           | n -> is_even (n - 1)
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  ```
- mutually recursive (btw: also tail recursive)

Parameter Accumulation

Why tail recursive functions?

Can be automatically transformed into space-efficient loops

Idea

- make function tail recursive
- provide data as input (additional parameter, often 'acc')
- computation *before* recursive call

Example (Range)

- ```
  let rec range m n = if m >= n then []
                                else m::range (m+1) n
  ```
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  let range_tl m n =
   let rec range acc m n =
    if m >= n then acc else range ((n-1)::acc) m (n-1)
   in
   range [] m n
  ```

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  let sumlen_fold xs =
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