

Functional Programming

WS 2016/17

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week 12



Overview

- Week 12 - Laziness
 - Summary of Weeks 10 & 11
 - Lazy Lists
 - Fibonacci Numbers
 - The Sieve of Eratosthenes



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Type Checking

- prove that expression e has a type τ w.r.t. environment E
- formally: $E \vdash e : \tau$
- use the inference rules of \mathcal{C} to do so

Type Inference

- find most general type $\alpha_0\sigma$ for expression e w.r.t. environment E
- formally: $E \triangleright e : \alpha_0$
- task is split into two parts:
 1. transform given type inference problem into unification problem
 2. solve the unification problem (result is substitution σ)

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This Week

Practice I

OCaml introduction, lists, strings, trees

Theory I

lambda-calculus, evaluation strategies, induction, reasoning about functional programs

Practice II

efficiency, tail-recursion,

Theory II

type checking, type inference

Advanced Topics

lazy evaluation, infinite data structures, combinator-parsing, dependent types, monads

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Motivation

Idea

Only compute values that are needed for the final result.

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Only compute values that are needed for the final result.

Example

In the program

```
let rec f x = if x = 0 then 0 else f x in
Lst.hd(f 0 :: f 1 :: [])
```

the value of 'f 1' is not needed. Nevertheless, the whole program does not terminate.

Custom Lazy Lists – 1st Iteration

Type

```
type 'a llist = Nil | Cons of ('a * 'a llist)
```

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```

Example

| | |
|---------------------|---------|
| Nil | ([]) |
| Cons(1,Nil) | ([1]) |
| Cons(2,Cons(1,Nil)) | ([2;1]) |

Custom Lazy Lists – 1st Iteration

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```

Example

| | |
|---------------------|---------|
| Nil | ([]) |
| Cons(1,Nil) | ([1]) |
| Cons(2,Cons(1,Nil)) | ([2;1]) |

Functions

```
let hd = function Nil      -> failwith "empty_list"
             | Cons(x,_) -> x

let rec from n = Cons(n,from(n+1))
```

Custom Lazy Lists – 1st Iteration (cont'd)

Problem

```
# hd(from 0);;  
Stack overflow ...
```

Custom Lazy Lists – 1st Iteration (cont'd)

Problem

```
# hd(from 0);;  
Stack overflow ...
```

Idea

- block computation of *tail*, until explicitly requested

Custom Lazy Lists – 1st Iteration (cont'd)

Problem

```
# hd(from 0);;  
Stack overflow ...
```

Idea

- block computation of *tail*, until explicitly requested
- use `unit` function (i.e., of type `unit -> ...`)

Custom Lazy Lists – 2nd Iteration (module UnitList)

Type

```
type 'a llist = Nil | Cons of ('a * (unit -> 'a llist))
```

Custom Lazy Lists – 2nd Iteration (module UnitList)

Type

```
type 'a llist = Nil | Cons of ('a * (unit -> 'a llist))
```

Example

```
Nil                                     ([])  
Cons(1, fun () -> Nil)                 ([1])  
Cons(2, fun () -> Cons(1, fun () -> Nil)) ([2;1])
```

Custom Lazy Lists – 2nd Iteration (module UnitList)

Type

```
type 'a llist = Nil | Cons of ('a * (unit -> 'a llist))
```

Example

```
Nil                                     ([])
Cons(1, fun () -> Nil)                 ([1])
Cons(2, fun () -> Cons(1, fun () -> Nil)) ([2;1])
```

Functions

```
let hd = function Nil          -> failwith "empty_list"
              | Cons(x,_)     -> x

let tl = function Nil          -> failwith "empty_list"
              | Cons(_,xs)    -> xs ()

let rec from n = Cons(n,fun() -> from(n+1))
```

Custom Lazy Lists – 2nd Iteration (cont'd)

Now

```
# hd(from 0);;  
- : int = 0  
  
# hd(tl(from 0));;  
- : int = 1
```

Custom Lazy Lists – 2nd Iteration (cont'd)

Now

```
# hd(from 0);;  
- : int = 0  
  
# hd(tl(from 0));;  
- : int = 1
```

But

- strange that *tail* of `l1ist` is not `l1ist` itself

Custom Lazy Lists – 2nd Iteration (cont'd)

Now

```
# hd(from 0);;
- : int = 0

# hd(tl(from 0));;
- : int = 1
```

But

- strange that *tail* of `llist` is not `llist` itself
- use a mutually recursive type

Custom Lazy Lists – 3rd Iteration (module UnitList)

Type

```
type 'a cell = Nil | Cons of ('a * 'a llist)
and 'a llist = (unit -> 'a cell)
```

Custom Lazy Lists – 3rd Iteration (module UnitList)

Type

```
type 'a cell = Nil | Cons of ('a * 'a llist)
and 'a llist = (unit -> 'a cell)
```

Example

```
fun () -> Nil                ([])
fun () -> Cons(1,fun () -> Nil) ([1])
fun () -> Cons(2,fun () -> Cons(1,fun () -> Nil)) ([2;1])
```


Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil      -> failwith "empty"
                | Cons(x,_) -> x
```

Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil      -> failwith "empty"
                | Cons(x,_) -> x

let rec from n = fun() -> Cons(n,from(n+1))
```

Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil          -> failwith "empty"
              | Cons(x,_) -> x

let rec from n = fun() -> Cons(n,from(n+1))

let rec to_list n xs = if n < 1 then [] else match xs() with
  | Nil          -> []
  | Cons(x,xs) -> x :: to_list (n-1) xs
```

Custom Lazy Lists – 3rd Iteration (module UnitList cont'd)

Functions

```
let hd xs = match xs() with Nil          -> failwith "empty"
                | Cons(x,_) -> x

let rec from n = fun() -> Cons(n,from(n+1))

let rec to_list n xs = if n < 1 then [] else match xs() with
| Nil          -> []
| Cons(x,xs) -> x :: to_list (n-1) xs
```

Example

```
# from 0;;
- : int llist = <fun>

# to_list 10 (from 0);;
- : int list = [0; 1; 2; 3; 4; 5; 6; 7; 8; 9]
```

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Recall

Definition (i -th Fibonacci number F_i)

$$F_i = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ F_{i-1} + F_{i-2} & \text{otherwise} \end{cases}$$

Recall

Definition (i -th Fibonacci number F_i)

$$F_i = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ F_{i-1} + F_{i-2} & \text{otherwise} \end{cases}$$

Sequence

0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 ...

Idea

Visualization

starting at 0 | 0 1

Idea

Visualization

| | | | |
|---------------|--|---|---|
| starting at 0 | | 0 | 1 |
| starting at 1 | | 1 | |

Idea

Visualization

| | | | |
|---------------|--|---|---|
| starting at 0 | | 0 | 1 |
| starting at 1 | | 1 | |
| (+) | | | |

Idea

Visualization

| | | | |
|---------------|--|---|---|
| starting at 0 | | 0 | 1 |
| starting at 1 | | 1 | |
| (+) | | | |

Idea

Visualization

| | | | |
|---------------|--|---|---|
| starting at 0 | | 0 | 1 |
| starting at 1 | | 1 | |
| (+) | | 1 | |

Idea

Visualization

| | | | | |
|---------------|--|---|---|---|
| starting at 0 | | 0 | 1 | 1 |
| starting at 1 | | 1 | 1 | |
| (+) | | 1 | | |

Idea

Visualization

| | | | | |
|---------------|--|---|---|---|
| starting at 0 | | 0 | 1 | 1 |
| starting at 1 | | 1 | 1 | |
| (+) | | 1 | | |

Idea

Visualization

| | | | | |
|---------------|--|---|---|---|
| starting at 0 | | 0 | 1 | 1 |
| starting at 1 | | 1 | 1 | |
| (+) | | 1 | 2 | |

Idea

Visualization

| | | | | | |
|---------------|--|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 |
| starting at 1 | | 1 | 1 | 2 | |
| (+) | | 1 | 2 | | |

Idea

Visualization

| | | | | | |
|---------------|--|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 |
| starting at 1 | | 1 | 1 | 2 | |
| (+) | | 1 | 2 | | |

Idea

Visualization

| | | | | | |
|---------------|--|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 |
| starting at 1 | | 1 | 1 | 2 | |
| (+) | | 1 | 2 | 3 | |

Idea

Visualization

| | | | | | | |
|---------------|--|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 |
| starting at 1 | | 1 | 1 | 2 | 3 | |
| (+) | | 1 | 2 | 3 | | |

Idea

Visualization

| | | | | | | |
|---------------|--|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 |
| starting at 1 | | 1 | 1 | 2 | 3 | |
| (+) | | 1 | 2 | 3 | | |

Idea

Visualization

| | | | | | | |
|---------------|--|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 |
| starting at 1 | | 1 | 1 | 2 | 3 | |
| (+) | | 1 | 2 | 3 | 5 | |

Idea

Visualization

| | | | | | | | |
|---------------|--|---|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | |
| (+) | | 1 | 2 | 3 | 5 | | |

Idea

Visualization

| | | | | | | | |
|---------------|--|---|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | |
| (+) | | 1 | 2 | 3 | 5 | | |

Idea

Visualization

| | | | | | | | |
|---------------|--|---|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | |

Idea

Visualization

| | | | | | | | | |
|---------------|--|---|---|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | | |

Idea

Visualization

| | | | | | | | | |
|---------------|--|---|---|---|---|---|---|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | | |

Idea

Visualization

| | | | | | | | | |
|---------------|--|---|---|---|---|---|----|---|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | |

Idea

Visualization

| | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | | |

Idea

Visualization

| | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | | |

Idea

Visualization

| | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | 21 | |

Idea

Visualization

| | | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | 21 | | |

Idea

Visualization

| | | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | 21 | | |

Idea

Visualization

| | | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | |

Idea

Visualization

| | | | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|-----|-----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | ... |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | ... | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | ... | |

Idea

Visualization

| | | | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|-----|-----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | ... |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | ... | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | ... | |

Missing

- function to shift sequence to the left
- function to add two sequences

Idea

Visualization

| | | | | | | | | | | | |
|---------------|--|---|---|---|---|---|----|----|----|-----|-----|
| starting at 0 | | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | ... |
| starting at 1 | | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | ... | |
| (+) | | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | ... | |

Missing

- function to shift sequence to the left → `tl`
- function to add two sequences → `zip_with (+)`

Implementation (in module UnitList)

```
let tl xs = match xs() with Nil          -> failwith "empty"
                | Cons(_,xs) -> xs

let rec zip_with f xs ys = fun() -> match (xs(),ys()) with
  | (Cons(x,xs),Cons(y,ys)) -> Cons(f x y,zip_with f xs ys)
  | _                        -> Nil

let rec fibs =
  fun() -> Cons(0,fun() -> Cons(1, zip_with (+) fibs (tl fibs)))
```

Implementation (in module UnitList)

```
let tl xs = match xs() with Nil          -> failwith "empty"
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let rec zip_with f xs ys = fun() -> match (xs(),ys()) with
  | (Cons(x,xs),Cons(y,ys)) -> Cons(f x y,zip_with f xs ys)
  | _                        -> Nil

let rec fibs =
  fun() -> Cons(0,fun() -> Cons(1, zip_with (+) fibs (tl fibs)))
```

Example

```
# to_list 10 fibs
- : int list = [0; 1; 1; 2; 3; 5; 8; 13; 21; 34]
```

Problem

Lazy Enough?

- we defer computation (i.e., call-by-name evaluation) ✓

Problem

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- we do not use **memoization** ✗

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Memoization

- prohibit recomputation of same expressions

Problem

Lazy Enough?

- we defer computation (i.e., call-by-name evaluation) ✓
- we do not use memoization ✗

Memoization

- prohibit recomputation of same expressions
- built-in in OCaml's support for laziness

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Lazyness in OCaml

Keyword `lazy`

used to transform arbitrary expression into **lazy** expression

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Example

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Example

- `let e = lazy(print_string "test\n")`

Lazyness in OCaml

Keyword `lazy`

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Example

- `let e = lazy(print_string "test\n")`
- `let f = lazy(let rec f() = print_int 1;f() in f())`

Lazyness in OCaml

Keyword `lazy`

used to transform arbitrary expression into **lazy** expression

Example

- `let e = lazy(print_string "test\n")`
- `let f = lazy(let rec f() = print_int 1;f() in f())`

Function `Lazy.force`

used to **evaluate** lazy expressions

Lazyness in OCaml

Keyword `lazy`

used to transform arbitrary expression into **lazy** expression

Example

- `let e = lazy(print_string "test\n")`
- `let f = lazy(let rec f() = print_int 1;f() in f())`

Function `Lazy.force`

used to **evaluate** lazy expressions

Example

- `Lazy.force e`

Lazyness in OCaml

Keyword `lazy`

used to transform arbitrary expression into **lazy** expression

Example

- `let e = lazy(print_string "test\n")`
- `let f = lazy(let rec f() = print_int 1;f() in f())`

Function `Lazy.force`

used to **evaluate** lazy expressions

Example

- `Lazy.force e`
- `Lazy.force f`

Lazy Lists Again (module LazyList)

Type

```
type 'a t    = 'a cell Lazy.t  
and 'a cell = Nil | Cons of ('a * 'a t)
```

Lazy Lists Again (module LazyList)

Type

```
type 'a t = 'a cell Lazy.t
and 'a cell = Nil | Cons of ('a * 'a t)
```

Example

```
lazy Nil                ([ ])
lazy (Cons(1, lazy Nil)) ([1])
lazy (Cons(2, lazy (Cons(1, lazy Nil)))) ([2;1])
```

The Sieve of Eratosthenes (module LazyList)

Algorithm

start with list of all natural numbers (from 2 on)

- ① mark first element h as prime
- ② remove all multiples of h
- ③ goto step ①

The Sieve of Eratosthenes (module LazyList)

Algorithm

start with list of all natural numbers (from 2 on)

- ① mark first element h as prime
- ② remove all multiples of h
- ③ goto step ①

Functions

```
let fc = Lazy.force
```

The Sieve of Eratosthenes (module LazyList)

Algorithm

start with list of all natural numbers (from 2 on)

- ① mark first element h as prime
- ② remove all multiples of h
- ③ goto step ①

Functions

```
let fc = Lazy.force
```

```
let rec from n = lazy(Cons(n,from(n+1)))
```

The Sieve of Eratosthenes (module LazyList)

Algorithm

start with list of all natural numbers (from 2 on)

- ① mark first element h as prime
- ② remove all multiples of h
- ③ goto step ①

Functions

```
let fc = Lazy.force
```

```
let rec from n = lazy(Cons(n,from(n+1)))
```

```
let rec to_list n xs = if n < 1 then [] else match fc xs with  
| Nil          -> []  
| Cons(x,xs)  -> x :: to_list (n-1) xs
```


The Sieve of Eratosthenes (module LazyList cont'd)

```
let rec filter p xs = lazy(match fc xs with
  | Nil          -> Nil
  | Cons(x,xs)  -> if p x then Cons(x,filter p xs)
                   else fc(filter p xs)
)
```

The Sieve of Eratosthenes (module LazyList cont'd)

```
let rec filter p xs = lazy(match fc xs with
  | Nil          -> Nil
  | Cons(x,xs)  -> if p x then Cons(x,filter p xs)
                   else fc(filter p xs)
)
```

```
let rec sieve xs = lazy(match fc xs with
  | Nil          -> Nil
  | Cons(x,xs)  ->
    Cons(x,sieve(filter (fun y -> y mod x <> 0) xs))
)
```

The Sieve of Eratosthenes (module LazyList cont'd)

```
let rec filter p xs = lazy(match fc xs with
  | Nil          -> Nil
  | Cons(x,xs)  -> if p x then Cons(x,filter p xs)
                   else fc(filter p xs)
)
```

```
let rec sieve xs = lazy(match fc xs with
  | Nil          -> Nil
  | Cons(x,xs)  ->
    Cons(x,sieve(filter (fun y -> y mod x <> 0) xs))
)
```

```
let primes = sieve(from 2)
```