Homework

In the LICS course (the book of Huth and Ryan) we have used natural deduction in 'box-style': proofs consists of *lists* of formulas, supplemented with (properly nested) box-structure. In this ITP course we instead use natural deduction in 'tree-style': proofs consists of *trees*, where the trees can be viewed as λ-terms. The exercises concern the correspondence between both, for the very simple case of the (intuitionistc) implicational fragment of propositional logic: the only connective is implication ($\rightarrow$) and we only have the respective introduction and elimination rules for $\rightarrow$

1. Give the introduction and elimination rules in both tree- and box-style.

2. Give proofs in both styles, tree and box, of:

$$\begin{array}{ll} \vdash & A \rightarrow A \\ \vdash & A \rightarrow B \rightarrow A \\ \vdash & (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C) \\ \vdash & A \rightarrow (A \rightarrow A \rightarrow B) \rightarrow (B \rightarrow B \rightarrow C) \rightarrow C \end{array}$$

   Which style do you prefer and why?

3. Give translations *tree* and *box* from box- to tree-style proofs (of the same formula), and conversely. Exemplify your translation for the third formula above.

   Is it the case that, for your translation, *tree* $\circ$ *box* is the identity, i.e. is it the case that translating a tree-style proof to a box-style proof and then back again yields the original proof? If so, argue why this is the case. If not, give an example showing this.

   The same question for *box* $\circ$ *tree*.

4. How can we see from a tree-style proof that it can be β-reduced? Give a tree-style proof of the fourth formula above that can be β-reduced, and β-reduce it. Define an analogon of β-reduction for box-style proofs, and exemplify it on the box-style proof of the fourth formula obtained by applying *box* to the tree-style proof above. Call that $\boxed{\beta}$. How does a box-style proof that cannot be $\boxed{\beta}$-reduced look like?

5. Give a series of provable formulas such that their proofs in $\beta/\boxed{\beta}$-normal form, in tree style are exponentially larger than their proofs in box style. Can you think of a modification of tree-style proofs that could overcome this problem?

6. Give the constructive interpretation of $A \rightarrow B$. (How) do proofs in tree- and and box-style proofs correspond to this interpretation?