Homework

1. Identify exactly where in the translation *tree*, mapping a box-style proof to a tree-style proof, the exponential blow-up occurs, for the formulas $A \to (A \to A \to B) \to (B \to B \to C) \to (C \to C \to D) \to ...$ as presented in the PS.

2. Install Coq

   `https://coq.inria.fr/download`

   and do the tutorial

   `https://coq.inria.fr/tutorial-nahas`

3. Prove $(A \to B \to C) \to (A \to B) \to (A \to C)$ in Coq in stepwise fashion (using intro and apply), and show in each step what is the goal, what are the assumptions, and what is the proof (in tree-style) constructed thus far (if you so wish, this proof may be represented by its λ-term instead).

4. (System F) We have the Church numerals $\underline{n} = \lambda fx.f^n x$ in the untyped λ-calculus, e.g. $\underline{1} = \lambda fx.fx$ and $\underline{2} = \lambda fx.f(fx)$. Addition, multiplication, and exponentiation of two numerals can be defined by the respective λ-terms $\lambda nmfx.nf(mfx)$, $\lambda nmfx.nf(mf)x$, $\lambda nm.nm$.

   - Show that these definitions are ok, by testing them on the inputs $\underline{1}$ and $\underline{2}$.

   The principal type of Church numerals in the simply typed λ-calculus is $(A \to A) \to (A \to A)$.

   - Show that when restriction to a fixed type A in the above, addition and multiplication can be typed appropriately, but exponentiaton cannot. Can you fix this (how?) when the restriction is dropped?

   In the polymorphic λ-calculus ($\lambda 2$) the Church numerals take a type as first (extra) parameter and are of type $(\forall A)(A \to A) \to (A \to A)$. In Coq it and addition can be defined accordingly by

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   Definition cnat := forall X : Type, (X -> X) -> X -> X.
   Definition addition  (n m: cnat) : cnat :=
   fun (X : Type)(f : X -> X)(x : X) => n X f (m X f x).
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   - Do the first item above in Coq, i.e. define multiplication and exponentiation and verify that they are well-defined, e.g. by proving that 2 two times 1 is 2.
   - Can you give the λ-terms in β-normal form inhabiting the type of Church numerals. Is each of them a natural number?

5. Prove in Coq the result on the correspondence between the square of sum and the sum of cubes (see the PS two weeks ago), again using elementary proof steps and lemmas. What are the differences between this proof and your earlier HOL Light proof, if any?