

Interactive Theorem Proving

Lecture 1

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Vincent van Oostrom (PS)

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Administration

Teacher

- Cezary Kaliszyk
- Consultation hours: Thursday afternoon, 3M12

Grading

- Lecture: Closed book exam
- Exercises: Assignments and participation
- Presentations

Practical Assignments

- Vincent van Oostrom
- Software: HOL Light, Mizar, Coq
- No RR this year → Laptops useful.

Lecture content

- Proof Assistants
- Simply typed lambda calculus
 - Church vs Curry, derivation formats, well-typedness, term finding
- Second-order typed lambda calculus
 - Π -types, second-order abstraction and application, $\lambda 2$
- Types dependent on types
 - Sorts, weakening, formation, properties
- Dependent types
 - λP , minimal logic, natural deduction again
- CoC
 - λ -cube, Girard's paradox, classical logic
- Definitions
 - terms, types, δ -conversion, \rightarrow_{Δ} , axioms
- Sets and set theory
- Numbers and arithmetic
 - \mathbb{N} , bits, efficient computation, divisibility, proof irrelevance

Proseminar content

- HOL Light introduction
- Kernel, rules, subgoal-package, tactics
- Type introduction, quotients, inductive
- Exercises for λP , $\lambda 2$
- Curry-Howard, BHK
- Logical Frameworks (LF, Pure)
- Proving properties modulo α
- Presentations: Automath, Minlog, Agda, Lego, PVS, ?

What is a Proof Assistant? (1/2)

A Proof Assistant is a

- a computer program
- to assist a mathematician
- in the production of a proof
- that is mechanically checked

What does a Proof Assistant do?

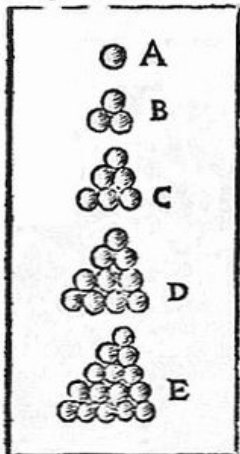
- Keep track of theories, definitions, assumptions
- Interaction - proof editing
- Proof checking
- Automation - proof search

What does it implement? (And how?)

- a formal logical system intended as foundation for mathematics
- decision procedures

The Kepler Conjecture (year 1611)

num pro apice, esto & alia cop

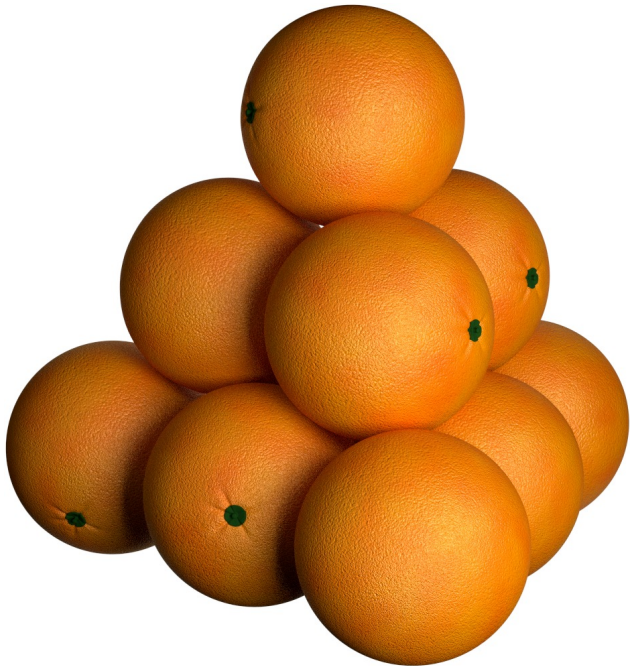


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The most compact way of stacking balls of the same size in space is a pyramid.

$$V = \frac{\pi}{\sqrt{18}} \approx 74\%$$

necessitate concurrente cum ra



The Kepler Conjecture (year 1611)

Proved in 1998

- Tom Hales, 300 page proof using computer programs
- Submitted to the Annals of Mathematics

The Kepler Conjecture (year 1611)

Proved in 1998

- Tom Hales, 300 page proof using computer programs
- Submitted to the Annals of Mathematics
- 99% correct... but we cannot verify the programs

1039 equalities and inequalities

For example:

$$\frac{-x_1 x_3 - x_2 x_4 + x_1 x_5 + x_3 x_6 - x_5 x_6 + x_2(-x_2 + x_1 + x_3 - x_4 + x_5 + x_6)}{\sqrt{4x_2 \left(\begin{array}{l} x_2 x_4 (-x_2 + x_1 + x_3 - x_4 + x_5 + x_6) + \\ + x_1 x_5 (x_2 - x_1 + x_3 + x_4 - x_5 + x_6) + \\ + x_3 x_6 (x_2 + x_1 - x_3 + x_4 + x_5 - x_6) - \\ - x_1 x_3 x_4 - x_2 x_3 x_5 - x_2 x_1 x_6 - x_4 x_5 x_6 \end{array} \right)}} < \tan\left(\frac{\pi}{2} - 0.74\right)$$

The Kepler Conjecture (year 1611)

Solution? Formalized Proof!

- Formalize the proof using Proof Assistants
- Implement the computer code in the system
- Prove the code correct
- Run the programs inside the Proof Assistant

Flyspeck Project

- Completed last year
- Many Proof Assistants and contributors

Intel Pentium P5 (1994)

FPU unit

- Division lookup table
- For certain inputs division result off

Replacement

- Few customers cared, still 450M\$
- Birth of HOL Light
- Intel and AMD processors **formally verified**

```

theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof
  assume "sqrt (real 2) ∈ ℚ"
  then obtain m n :: nat where
    n_nonzero: "n ≠ 0" and sqrt_rat: "√(real 2) = real m / real n"
    and lowest_terms: "gcd m n = 1" ..
  from n_nonzero and sqrt_rat have "real m = √(real 2) * real n" by simp
  then have "real (m2) = (sqrt (real 2))2 * real (n2)"
    by (auto simp add: power2_eq_square)
  also have "(sqrt (real 2))2 = real 2" by simp
  also have "... * real (m2) = real (2 * n2)" by simp
  finally have eq: "m2 = 2 * n2" ..
  hence "2 dvd m2" ..
  with two_is_prime have dvd_m: "2 dvd m" by (rule prime_dvd_power_two)
  then obtain k where "m = 2 * k" ..
  with eq have "2 * n2 = 22 * k2" by (auto simp add: power2_eq_square mult_ac)
  hence "n2 = 2 * k2" by simp
  hence "2 dvd n2" ..
  with two_is_prime have "2 dvd n" by (rule prime_dvd_power_two)
  with dvd_m have "2 dvd gcd m n" by (rule gcd_greatest)
  with lowest_terms have "2 dvd 1" by simp
  thus False by arith
qed

```

```

theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof
  assume "sqrt (real 2) ∈ ℚ"
  then obtain m n :: nat where
    n_nonzero: "n ≠ 0" and sqrt_rat: "√(real 2) = real m / real n"
    and lowest_terms: "gcd m n = 1" ..
  from n_nonzero and sqrt_rat have "real m = √(real 2) * real n" by simp
  then have "real (m²) = (sqrt (real 2))² * real (n²)"
    by (auto simp add: power2_eq_square)
  also have "(sqrt (real 2))² = real 2" by simp
  also have "... * real (m²) = real (2 * n²)" by simp
  finally have eq: "m² = 2 * n²" ..
  hence "2 dvd m²" ..
  with two_is_prime have dvd_m: "2 dvd m" by (rule prime_dvd_power_two)
  then obtain k where "m = 2 * k" ..
  with eq have "2 * n² = 2² * k²" by (auto simp add: power2_eq_square mult_ac)
  hence "n² = 2 * k²" by simp
  hence "2 dvd n²" ..
  with two_is_prime have "2 dvd n" by (rule prime_dvd_power_two)
  with dvd_m have "2 dvd gcd m n" by (rule gcd_greatest)
  with lowest_terms have "2 dvd 1" by simp
  thus False by arith
qed

```

de Bruijn factor

:: W The Irrationality of the Square Root of 2

theorem Th1:

for p being Element of NAT st p is prime holds
sqrt p is irrational

proof

```
let p be Element of NAT ;
assume A1: p is prime ;
then A2: p > 1 by INT_2:def 4;
assume sqrt p is rational ;
then consider i being Integer, n being Element of NAT such that
A3: n <> 0 and
A4: sqrt p = i / n and
A5: for i1 being Integer
for n1 being Element of NAT st n1 <> 0 & sqrt p = i1 / n1 holds
n <= n1 by RAT_1:9;
A6: i = (sqrt p) * n by A3, A4, XCMPLEX_1:87;
sqrt p >= 0 by SQUARE_1:def 2;
then reconsider m = i as Element of NAT by A6, INT_1:3;
A7: m ^2 = ((sqrt p) ^2) * (n ^2) by A6
.= p * (n ^2) by SQUARE_1:def 2 ;
then p divides m ^2 by NAT_D:def 3;
then p divides m by A1, NEWTON:80;
then consider m1 being Nat such that
A8: m = p * m1 by NAT_D:def 3;
n ^2 = (p * (p * (m1 ^2))) / p by A2, A7, A8, XCMPLEX_1:89
.= p * (m1 ^2) by A2, XCMPLEX_1:89 ;
then p divides n ^2 by NAT_D:def 3;
then p divides n by A1, NEWTON:80;
then consider n1 being Nat such that
A9: n = p * n1 by NAT_D:def 3;
A10: m1 / n1 = sqrt p by A2, A4, A8, A9, XCMPLEX_1:91;
A11: n1 <> 0 by A3, A9;
then p * n1 > 1 * n1 by A2, XREAL_1:98;
hence contradiction by A5, A9, A11, A10;
end;
```

Proof Assistant (2/2)

- Keep track of theories, definitions, assumptions
 - set up a theory that describes mathematical concepts (or models a computer system)
 - express logical properties of the objects
- Interaction - proof editing
 - typically interactive
 - specified theory and proofs can be edited
 - provides information about required proof obligations
 - allows further refinement of the proof
 - often manually providing a direction in which to proceed.
- Automation - proof search
 - various strategies
 - decision procedures
- Proof checking
 - checking of complete proofs
 - sometimes providing certificates of correctness
- Why should we trust it?
 - small core

Can a Proof Assistant do all proofs?

Decidability!

- Validity of formulas is undecidable
- (for non-trivial logical systems)

Automated Theorem Provers

- Specific domains
- Adjust your problem
- Answers: Valid (Theorem with proof)
- Or: Countersatisfiable (Possibly with counter-model)

Proof Assistants

- Generally applicable
- Direct modelling of problems
- Interactive

Other Tools

Computer Algebra

- Solving equations, simplifications, numerical approximations
- Maple, Mathematica, ...

Model Checkers

- Space state abstraction
- Spin, Uppaal, ...

ATPs

- Built in automation (model elimination, resolution)
- ACL2, Vampire, Eprover, SPASS, ...

Users of Proof Assistants

Computer Science

- Modelling and specifying systems
- Proving properties of systems
- Proving software correct

Mathematics

- Defining concepts and theories
- Proving (mostly verifying) proofs
- (currently less common)

Theorems and programs that use ITP

Theorems

- Kepler Conjecture (2014)
- 4 color theorem
- Feit-Thomson theorem (2012)

Software

- Processors and Chips
- Security Protocols
- Project Cristal (Comp-Cert)
- L4-Verified
- Java Bytecode

History of Proof Assistants

λ -calculus (Church, 1940)

- Simple Type Theory
- Higher-Order Logic

Formulas as Types (Curry-Howard, de Bruijn)

- Proofs as Terms
- Reduce Proof Checking to Type Checking

Automath

- First implementation

LCF (Milner)

- ML programming language

Multitude of Proof Assistants

Characterized by various

- Foundations
- Interaction models
- Automation strategies
- Libraries
- Size of trusted core

Examples

- HOL (HOL4, HOL-Light, ProofPower, HOL0), Mizar (and variants), PVS, Coq, Otter/Ivy, Isabelle/Isar (HOL, ZF, CTT, ...), Alfa/Agda, ACL2, IMPS, Metamath, Theorema, Lego, Nuprl, Ω mega, B method, Minlog

Coverage of Basic Mathematics

Freek Wiedijk's list of 100 theorems

HOL Light	86
Isabelle	70
Mizar	64
Coq	63
MetaMath	58
ProofPower	42
nqthm/ACL2	18
PVS	16
NuPRL/MetaPRL	8
any	92

<http://www.cs.ru.nl/~freek/100/>

Summary

This Lecture

- What is a Proof Assistant
- Common Uses
- Comparison with other tools
- Formal proof examples
- De Bruijn factor
- History
- Characteristics
- Coverage of Basic Mathematics

Next

- Typed λ -calculus
- HOL Light