Computational Logic

Interactive Theorem Proving

Week 7

Cezary Kaliszyk
Vincent van Oostrom

November 18, 2016

$\lambda 2$

$\lambda 2$

(start-rule) $\dfrac{(x{:}A) \in \Gamma}{\Gamma \vdash x : A}$,

($\to$-elimination) $\dfrac{\Gamma \vdash M : (A \to B) \quad \Gamma \vdash N : A}{\Gamma \vdash (M N) : B}$;

($\to$-introduction) $\dfrac{\Gamma, a{:}A \vdash M : B}{\Gamma \vdash (\lambda a{:}A.M) : (A \to B)}$;

($\forall$-elimination) $\dfrac{\Gamma \vdash M : (\forall \alpha.A)}{\Gamma \vdash M B : A[\alpha := B]}$, $B \in \mathbb{T}$;

($\forall$-introduction) $\dfrac{\Gamma \vdash M : A}{\Gamma \vdash (\Lambda \alpha.M) : (\forall \alpha.A)}$, $\alpha \notin \mathrm{FV}(\Gamma)$.

$$(\lambda a{:}A.M)N \to_\beta M[a := N]$$
$$(\Lambda \alpha.M)A \to_\beta M[\alpha := A]$$

$\lambda 2$ examples

derivation

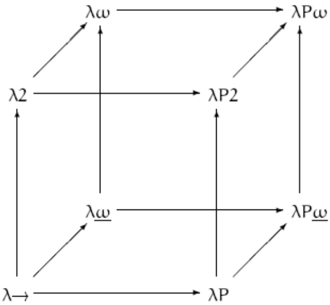
$$
\begin{array}{rl}
 & \vdash (\lambda a{:}\alpha.a) : (\alpha{\rightarrow}\alpha); \\
 & \vdash (\Lambda\alpha\lambda a{:}\alpha.a) : (\forall\alpha.\alpha{\rightarrow}\alpha); \\
 & \vdash (\Lambda\alpha\lambda a{:}\alpha.a)A : (A{\rightarrow}A); \\
b{:}A & \vdash (\Lambda\alpha\lambda a{:}\alpha.a)Ab : A;
\end{array}
$$

reduction

$$(\Lambda\alpha\lambda a{:}\alpha.a)Ab{\rightarrow}(\lambda a{:}A.a)b{\rightarrow}b; \}$$

$$\vdash (\Lambda\beta\lambda a{:}(\forall\alpha.\alpha).a((\forall\alpha.\alpha){\rightarrow}\beta)a) : (\forall\beta.(\forall\alpha.\alpha){\rightarrow}\beta);$$

λ-cube cube



λ-cube

Systems in the λ-cube

1. General axiom and rules.

(axiom) $<> \vdash * : \square;$

(start rule) $$\frac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A}, \; x \notin \Gamma;$$

(weakening rule) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x{:}C \vdash A : B}, \; x \notin \Gamma;$$

(application rule) $$\frac{\Gamma \vdash F : (\Pi x{:}A.B) \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x := a]};$$

(abstraction rule) $$\frac{\Gamma, x{:}A \vdash b : B \quad \Gamma \vdash (\Pi x{:}A.B) : s}{\Gamma \vdash (\lambda x{:}A.b) : (\Pi x{:}A.B)};$$

(conversion rule) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B =_\beta B'}{\Gamma \vdash A : B'}.$$

2. The specific rules

(s_1, s_2) rule $$\frac{\Gamma \vdash A : s_1, \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\Pi x{:}A.B) : s_2}.$$

λ-cube dependencies

System	Set of specific rules			
$\lambda\to$	$(*,*)$			
$\lambda 2$	$(*,*)$	$(\Box,*)$		
λP	$(*,*)$		$(*,\Box)$	
$\lambda P2$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	
$\lambda\underline{\omega}$	$(*,*)$			$(\Box,\Box)$
$\lambda\omega$	$(*,*)$	$(\Box,*)$		$(\Box,\Box)$
$\lambda P\underline{\omega}$	$(*,*)$		$(*,\Box)$	$(\Box,\Box)$
$\lambda P\omega = \lambda C$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	$(\Box,\Box)$

$\lambda 2$ cube examples

$$\begin{array}{rcl}
\alpha{:}* & \vdash & (\lambda a{:}\alpha.a) : (\alpha \to \alpha); \\
 & \vdash & (\lambda \alpha{:}*\lambda a{:}\alpha.a) : (\Pi \alpha{:}*.(\alpha \to \alpha)) : *; \\
A{:}* & \vdash & (\lambda \alpha{:}*\lambda a{:}\alpha.a)A : (A \to A); \\
A{:}*, b{:}A & \vdash & (\lambda \alpha{:}*\lambda a{:}\alpha.a)Ab : A;
\end{array}$$

reduction

$$\begin{array}{rcl}
(\lambda \alpha{:}*\lambda a{:}\alpha.a)Ab & \to & (\lambda a{:}A.a)b \\
 & \to & b.
\end{array}$$

$\lambda\!\rightarrow$ and $\lambda 2$ in λ cube

$$A\!\rightarrow\! B \equiv \Pi x{:}A.B, \text{ where } x \text{ is fresh (not in } A, B).$$

$$A{:}*, B{:}*, a{:}A, b{:}B \vdash M : C : *$$

$$a{:}A, b{:}B \vdash M : C$$

$$\forall\alpha.A \equiv \Pi\alpha{:}*.A,$$

$$\Lambda\alpha.M \equiv \lambda\alpha{:}*.M.$$

Semantics of $\lambda 2$?

- What is meaning of $\forall \alpha . \alpha \to \alpha$?

Semantics of $\lambda 2$?

- What is meaning of $\forall\alpha.\alpha \to \alpha$?
- instantiate α by *all* possible types ...

Semantics of $\lambda 2$?

- What is meaning of $\forall \alpha . \alpha \to \alpha$?
- instantiate α by *all* possible types ...
- ...including $\forall \alpha . \alpha \to \alpha$ itself?

Semantics of $\lambda 2$?

- What is meaning of $\forall \alpha . \alpha \to \alpha$?
- instantiate α by *all* possible types ...
- ... including $\forall \alpha . \alpha \to \alpha$ itself?
- impredicative

Semantics of $\lambda 2$?

- What is meaning of $\forall \alpha . \alpha \to \alpha$?
- instantiate α by *all* possible types ...
- ... including $\forall \alpha . \alpha \to \alpha$ itself?
- impredicative

Russell: set of all sets that are not members of themselves.

Semantics of $\lambda 2$?

- What is meaning of $\forall \alpha . \alpha \to \alpha$?
- instantiate α by *all* possible types ...
- ... including $\forall \alpha . \alpha \to \alpha$ itself?
- impredicative

Russell: set of all sets that are not members of themselves.
least upper bound

Semantics of λ-calculi

- $\lambda\!\!\rightarrow$
- untyped λ-calculus
- $\lambda 2$

Strong normalisation of $\lambda 2$

1. $\mathrm{SN} = \{M \in \Lambda \mid M \text{ is strongly normalizing}\}$.
2. Let $A, B \subseteq \Lambda$. Define $A \rightarrow B$ a subset of Λ by
$$A \rightarrow B = \{F \in \Lambda \mid \forall a \in A \; Fa \in B\}.$$
3. For every $\sigma \in \mathrm{Type}(\lambda\rightarrow)$ a set $[\![\sigma]\!] \subseteq \Lambda$ is defined as follows:
$$\begin{array}{rcl} [\![\alpha]\!] & = & \mathrm{SN}, \text{ where } \alpha \text{ is a type variable;} \\ [\![\sigma \rightarrow \tau]\!] & = & [\![\sigma]\!] \rightarrow [\![\tau]\!]. \end{array}$$

1. A subset $X \subseteq \mathrm{SN}$ is called *saturated* if
   (a) $\forall n \geq 0 \, \forall R_1, \ldots, R_n \in \mathrm{SN} \;\; x\vec{R} \in X$,
   where x is any term variable;
   (b) $\forall n \geq 0 \, \forall R_1, \ldots, R_n \in \mathrm{SN} \forall Q \in \mathrm{SN}$
   $$P[x := Q]\vec{R} \in X \quad \Rightarrow \quad (\lambda x.P)Q\vec{R} \in X.$$
2. $\mathrm{SAT} = \{X \subseteq \Lambda \mid X \text{ is saturated}\}$.

Strong normalisation of $\lambda 2$

1. $\mathrm{SN} \in \mathrm{SAT}$.
2. $A, B \in \mathrm{SAT} \Rightarrow A \rightarrow B \in \mathrm{SAT}$.
3. *Let* $\{A_i\}_{i \in I}$ *be a collection of members of* SAT, *then* $\bigcap_{i \in I} A_i \in$
4. *For all* $\sigma \in$ *Type*$(\lambda\rightarrow)$ *one has* $[\![\sigma]\!] \in \mathrm{SAT}$.

Parametric polymorphism

- Ad hoc vs. parametric polymorphism
- Theorems for free
- $\forall A.(A \to B) \to (A^* \to B^*)$?