Logic Programming

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Summary of Last Lecture

Example (meta-variable facility)

```
X; Y :- X.
X; Y :- Y.
```

Definitions (second-order programming)

- the predicate *bagof*(*Template*,*Goal*,*Bag*) unifies *Bag* with the alternatives of *Template* that meet *Goal*
- if *Goal* has free variables besides the one sharing with *Template* *bagof* will backtrack
- fails if *Goal* has no solutions
- construct *Var*^*Goal* tells *bagof* to existentially quantify *Var*
- the predicate *setof*(*Template*,*Goal*,*Bag*) is similar to *bagof* but sorts the obtained multi-set (bag) and removed duplicates

Outline of the Lecture

Monotone Logic Programs

introduction, basic constructs, logic foundations, unification, semantics, database and recursive programming, termination, complexity

Incomplete Data Structures and Constraints

incomplete data structures, definite clause grammars, constraint logic programming, answer set programming

Full Prolog

semantics (revisted), cuts, correctness proofs, meta-logical predicates, pragmatics, efficient programs, meta programming

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Example

```
sublist(Xs,AXBs) :- suffix(XBs,AXBs), prefix(Xs,XBs).
sublist(Xs,AXBs) :- prefix(AXs,AXBs), suffix(Xs,AXs).
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Question

What is better, if we argue wrt a linked-list implementation of cons lists?

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What is better, if we argue wrt a linked-list implementation of cons lists?

Answer

the first alternative:

- consider

  ```
  sublist([1,2,3,4],[1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4])
  ```

- the 1st clause iterates over the 2nd list to find a suitable suffix
- then iterates over the first list
- no intermediate data structures are created

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  sublist([1,2,3,4],[1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4])
  ```

- the 1st clause iterates over the 2nd list to find a suitable suffix
- then iterates over the first list
- no intermediate data structures are created
- in the 2nd clause an auxilliary list is created

Definition

we say: the first clause doesn't cons

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Observations on Time

- if full unification (unification of two arbitrary terms in goals) is not employed, reduction of a goal using a clause needs constant time
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- equivalently the number of unifications (performed and attempted) asymptotically bounds the runtime
- on the other hand, if unification needs to be taken into account time complexity analysis is more involved
- in general size of search space and size of input terms needs to be taken into account

Howto Improve Performance

Suggestion ①

use better algorithms ☺

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Example

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reverse([X|Xs],Zs) :-
    reverse(Xs,Ys),
    append(Ys,[X],Zs).
reverse([],[]).
```

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Example

```
reverse(Xs,Ys) :- reverse(Xs,[],Ys).
reverse([X|Xs],Acc,Ys) :-
    reverse(Xs,[X|Acc],Ys).
reverse([],Ys,Ys).
```

Excursion: Transforming Recursion into Iteration

Definitions

- a Prolog clause is called iterative if
  1. it has one recursive call, and
  2. zero or more calls to system predicates, before the recursive call
- a Prolog procedure is iterative if it contains only facts and iterative clauses

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Example (Factorial Iterative, Version 1)

```
factorial(N,F) :- factorial(0,N,1,F).

factorial(I,N,T,F) :-
    I < N, I1 is I + 1, T1 is T*I1, factorial(I1,N,T1,F).
factorial(N,N,F,F).
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Example (Factorial Iterative, Version 2)

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factorial(0,F,F).
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between(I,J,I) :- I ⩽ J.
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Example

```
sumlist(Is,Sum) :- sumlist(Is,0,Sum).

sumlist([I|Is],Temp,Sum) :-
    Temp1 is Temp + I,sumlist(Is,Temp1,Sum).
sumlist([],Sum,Sum).
```

Example

```
maximum([X|Xs],M) :- maximum(Xs,X,M).

maximum([X|Xs],Y,M) :-
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Example

```
length([X|Xs],N) :-
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Suggestion ②

tuning, via:

1. good goal order
2. elimination of (unwanted) nondeterminism by using explicit conditions and cuts
3. exploit clause indexing (order arguments suitably)
   indexing performs static analysis to detect clauses which are applicable for reduction

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Example

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append([X|Xs],Ys,[X|Zs]) :-
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By default, SWI-Prolog, as most other implementations, indexes predicates on their first argument.

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Definition (tail recursion optimisation)

- consider a generic clause for A

$$A' : -B_1, \ldots, B_n$$

  such that A and A' unify with σ
- suppose the goal $B_1\sigma, \ldots, B_{n-1}\sigma$ is deterministic
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Definition

clause indexing is used to detect which clauses are applicable for reduction: 2nd clause in append need only be considered for empty lists

How to Implement Functions

Functions vs Relations

- often, we want to compute functions:
  1. addition: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$
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- that is, we implement functions $f(i_1, \ldots, i_n) = (o_1, \ldots, o_m)$ by relations $f_{rel}/(n+m)$
- result is obtained by query $f_{rel}(i_1, \ldots, i_n, X_1, \ldots, X_m)$
  1. addition: $plus(n, m, Z)$ $\qquad Z = n + m$
  2. sorting: $sort(list, Xs)$ $\qquad Xs =$ sorted version of $list$

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f(X,Y) :- times(X,X,Z), minus(Z,5,V), times(7,V,U),
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Example (Ackermann function in Haskell)

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ack 0 m = m + 1
ack (n+1) m = if m == 0 then ack n 1 else
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Example (Ackermann function as logic program)

```
ack(0,M,s(M)).
ack(s(N),M,R) :- =(M,0,B), cond(B,N,M,R).
cond(true,N,M,R) :- ack(N,s(0),R).
cond(false,N,M,R) :- -(M,s(0),U),ack(s(N),U,V),ack(N,V,R).
```

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can be programmed as

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can be programmed as

```
f(X,Y) :- eval(s(X*X) - X*X, Y).
```

- evaluator is simple logic program (actually a simple meta interpretor)

```
eval(0,0).
eval(s(E),s(N)) :- eval(E,N).
eval(E+F,K) :- eval(E,N), eval(F,M), plus(N,M,K).
eval(E-F,K) :- eval(E,N), eval(F,M), plus(M,K,N).
eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

`f(s(s(0)),Y)`

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                          |
                     f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                   |
          eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                   |
                            f(s(s(0)),Y)
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Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                N = s(N1) |
 eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                          |
           eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                          |
                                  f(s(s(0)),Y)
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Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                 |
              eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                     N = s(N1) |
               eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                                 |
                        eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                 |
                                       f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                |
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                                                |
                      eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                |
                                        f(s(s(0)),Y)
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Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                |
            eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N = s(N1) |
             eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                                |
                      eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                |
                                        f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                |
        eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                        N = s(N1) |
         eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                                |
                  eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                |
                                        f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
                    times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                N3 = s(s(0)) ||
        eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                 N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                |
                    eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                N = s(N1) |
                     eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                                |
                              eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                |
                                                f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
                    eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(s(0))))))
                                  N1 = s(s(s(s(0)))) ||
          times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                    N3 = s(s(0)) ||
     eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                       N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                     N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                     N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                                |
           eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                      N = s(N1) |
            eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                                |
                     eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                                |
                                       f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
                    plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                    M = s(s(s(s(0)))) ||
          eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(s(0))))))
                    N1 = s(s(s(s(0)))) ||
   times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                    N3 = s(s(0)) ||
eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                    N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                    N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                    N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                |
   eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                    N = s(N1) |
    eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                |
          eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                |
                        f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
                                   □
                          Y = s(0) ‖
        plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                 M = s(s(s(s(0)))) ‖
   eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(s(0))))))
               N1 = s(s(s(s(0)))) ‖
 times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                     N3 = s(s(0)) ‖
 eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                           N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                         N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                         N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                    |
 eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                           N = s(N1) |
  eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                    |
          eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                    |
                          f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(s(X*X) - X*X, Y).`)

```
                                   □
                          Y = s(0) ‖
        plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                 M = s(s(s(s(0)))) ‖
    eval(s(s(0))*s(s(0)),M), plus(M,Y,s(s(s(s(s(0))))))
                N1 = s(s(s(s(0)))) ‖
  times(s(s(0)),s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                      N3 = s(s(0)) ‖
eval(s(s(0)),N3), times(s(s(0)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                            N5 = 0 |
eval(0,N5), eval(s(s(0)),N3), times(s(s(N5)),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                          N4 = s(N5) |
eval(s(0),N4), eval(s(s(0)),N3), times(s(N4),N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                          N2 = s(N4) |
eval(s(s(0)),N2), eval(s(s(0)),N3), times(N2,N3,N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                                     |
   eval(s(s(0))*s(s(0)),N1), eval(s(s(0))*s(s(0)),M), plus(M,Y,s(N1))
                            N = s(N1) |
    eval(s(s(s(0))*s(s(0))),N), eval(s(s(0))*s(s(0)),M), plus(M,Y,N)
                                     |
          eval(s(s(s(0))*s(s(0))) - s(s(0))*s(s(0)),Y)
                                     |
                          f(s(s(0)),Y)
```

Speeding up evaluation using "let"

- consider sub-expression `X*X`

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- `let(X,E,F)` encodes $let\ x = e\ in\ f$

```
eval(0,0).
eval(s(E),s(N)) :- eval(E,N).
eval(E+F,K) :- eval(E,N), eval(F,M), plus(N,M,K).
eval(E-F,K) :- eval(E,N), eval(F,M), plus(M,K,N).
eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
eval(let(X,E,F),K) :- eval(E,N), X = N, eval(F,K).
```

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eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
eval(let(X,E,F),K) :- eval(E,N), X = N, eval(F,K).
```

Example

```
f(X,Y) :- eval(s(X*X) - X*X, Y).
f(X,Y) :- eval(let(X2, X*X, s(X2) - X2), Y).
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                        |
                f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                |
     eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                |
                      f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
      N = s(s(s(s(0)))) ||
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                          |
eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                          |
                f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
        X2 = s(s(s(s(0)))) |
 X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
        N = s(s(s(s(0)))) ||
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                          |
   eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                          |
                  f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
eval(s(s(s(s(s(0))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                  |
           eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
                 X2 = s(s(s(s(0)))) |
            X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
                  N = s(s(s(s(0)))) ||
      eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                  |
          eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                  |
                        f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
      eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
                 N = s(s(s(s(s(0))))) ||
eval(s(s(s(s(s(0)))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                  |
           eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
                 X2 = s(s(s(s(0)))) |
            X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
                 N = s(s(s(s(0)))) ||
      eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                  |
          eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                  |
                        f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
        M = s(s(s(s(0)))) ||
eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
        N = s(s(s(s(s(0))))) ||
eval(s(s(s(s(s(0)))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                |
eval(s(s(s(s(s(0)))))-s(s(s(s(0)))),Y)
        X2 = s(s(s(s(0)))) |
X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
        N = s(s(s(s(0)))) ||
eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                |
eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                |
                        f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
                                  □
                        Y = s(0) ‖
        plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                M = s(s(s(s(0)))) ‖
   eval(s(s(s(s(0)))),M), plus(M,Y,s(s(s(s(s(0))))))
            N = s(s(s(s(s(0))))) ‖
eval(s(s(s(s(s(0)))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                  |
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              X2 = s(s(s(s(0)))) |
         X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
              N = s(s(s(s(0)))) ‖
   eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                  |
      eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                  |
                        f(s(s(0)),Y)
```

Example (`f(X,Y) :- eval(let(X2,X*X,s(X2)-X2), Y).`)

```
                                  □
                        Y = s(0) ‖
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                M = s(s(s(s(0)))) ‖
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eval(s(s(s(s(s(0)))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
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              X2 = s(s(s(s(0)))) |
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              N = s(s(s(s(0)))) ‖
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```
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                        Y = s(0) ‖
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            N = s(s(s(s(s(0))))) ‖
eval(s(s(s(s(s(0)))))),N), eval(s(s(s(s(0)))),M), plus(M,Y,N)
                                  |
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         X2 = s(s(s(s(0)))), eval(s(X2)-X2,Y)
              N = s(s(s(s(0)))) ‖
   eval(s(s(0))*s(s(0)),N), X2 = N, eval(s(X2)-X2,Y)
                                  |
      eval(let(X2,s(s(0))*s(s(0)),s(X2)-X2),Y)
                                  |
                        f(s(s(0)),Y)
```

Speeding up "let" even further

- detected problems:
  1. after computing x^2, result is evaluated again
     `eval(s(s(s(s(0)))),M)`
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eval(E*F,K) :- eval(E,N), eval(F,M), times(N,M,K).
eval(num(N),N).
eval(let(X,E,F),K) :- eval(E,N),X = num(N), eval(F,K).
```

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```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y)`)

```
f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y)`)

```
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                |
                        f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y)`)

```
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                |
                        f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                  |
       eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                    GX = num(s(s(0))) |
        GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                  |
                          f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
            eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
                   eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                                 GX = num(s(s(0))) |
                    GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                                |
                                        f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                    N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
        eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                                |
              eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                            GX = num(s(s(0))) |
               GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                                |
                                      f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                                N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                             |
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                             |
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                           GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                             |
                                   f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
N = s(s(s(s(0)))) ||
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
|
f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
        X2 = num(s(s(s(s(0))))) |
    X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
            N = s(s(s(s(0)))) ||
  times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                |
  eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                |
    eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
            GX = num(s(s(0))) |
    GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                |
                        f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
eval(s(num(s(s(s(s(0))))))),N), eval(num(s(s(s(s(0)))))),M), plus(M,Y,N)
                                        |
          eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
                  X2 = num(s(s(s(s(0))))) |
             X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                      N = s(s(s(s(0)))) ||
       times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                          N2 = s(s(0)) |
   eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                          N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                        |
      eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                        |
          eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                      GX = num(s(s(0))) |
          GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                        |
                              f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
eval(num(s(s(s(s(0))))),N1), eval(num(s(s(s(s(0))))),M), plus(M,Y,s(N1))
                                N = s(N1) |
 eval(s(num(s(s(s(s(0))))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
                                          |
            eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
                     X2 = num(s(s(s(s(0))))) |
                 X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                          N = s(s(s(s(0)))) ||
          times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                              N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                              N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                          |
     eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                          |
            eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                          GX = num(s(s(0))) |
            GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                          |
                                f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
eval(num(s(s(s(s(0))))),M), plus(M,Y,s(s(s(s(s(0))))))
N1 = s(s(s(s(0)))) |
eval(num(s(s(s(s(0))))),N1), eval(num(s(s(s(s(0))))),M), plus(M,Y,s(N1))
N = s(N1) |
eval(s(num(s(s(s(s(0))))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
|
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
X2 = num(s(s(s(s(0))))) |
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
N = s(s(s(s(0)))) ||
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
|
f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
M = s(s(s(s(0)))) |
eval(num(s(s(s(s(0))))),M), plus(M,Y,s(s(s(s(s(0))))))
N1 = s(s(s(s(0)))) |
eval(num(s(s(s(s(0))))),N1), eval(num(s(s(s(s(0))))),M), plus(M,Y,s(N1))
N = s(N1) |
eval(s(num(s(s(s(s(0))))))),N), eval(num(s(s(s(s(0))))),M), plus(M,Y,N)
|
eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0))))),Y)
X2 = num(s(s(s(s(0))))) |
X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
N = s(s(s(s(0)))) ||
times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
N2 = s(s(0)) |
eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
|
eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
GX = num(s(s(0))) |
GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
|
f(s(s(0)),Y)
```

Example (`f(X,Y):-GX=num(X),eval(let(X2,GX*GX,s(X2)-X2),Y))`

```
                                        □
                                Y = s(0) ||
                    plus(s(s(s(s(0)))),Y,s(s(s(s(s(0))))))
                            M = s(s(s(s(0)))) |
            eval(num(s(s(s(s(0)))))),M), plus(M,Y,s(s(s(s(s(0))))))
                           N1 = s(s(s(s(0)))) |
    eval(num(s(s(s(s(0)))))),N1), eval(num(s(s(s(s(0)))))),M), plus(M,Y,s(N1))
                                 N = s(N1) |
     eval(s(num(s(s(s(s(0))))))),N), eval(num(s(s(s(s(0)))))),M), plus(M,Y,N)
                                         |
                eval(s(num(s(s(s(s(0))))))-num(s(s(s(s(0)))))),Y)
                        X2 = num(s(s(s(s(0))))) |
                   X2 = num(s(s(s(s(0))))), eval(s(X2)-X2,Y)
                            N = s(s(s(s(0)))) ||
            times(s(s(0)),s(s(0)),N), X2 = num(N), eval(s(X2)-X2,Y)
                                 N2 = s(s(0)) |
    eval(num(s(s(0)),N2), times(s(s(0)),N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                 N1 = s(s(0)) |
eval(num(s(s(0)),N1), eval(num(s(s(0)),N2), times(N1,N2,N), X2 = num(N), eval(s(X2)-X2,Y)
                                         |
        eval(num(s(s(0)))*num(s(s(0))),N), X2 = num(N), eval(s(X2)-X2,Y)
                                         |
              eval(let(X2,num(s(s(0)))*num(s(s(0))),s(X2)-X2),Y)
                            GX = num(s(s(0))) |
              GX = num(s(s(0))), eval(let(X2,GX*GX,s(X2)-X2),Y)
                                         |
                                  f(s(s(0)),Y)
```