Computational Logic

Functional Programming

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WS 2017/2018

Lecture 5

Topics

abstract data types, algebraic data types, binary search trees, combinator parsing, efficiency, encoding data types as lambda-terms, evaluation strategies, formal verification, first steps, guarded recursion, Haskell introduction, higher-order functions, historical overview, implementing a type checker, induction, infinite data structures, input and output, lambda-calculus, lazy evaluation, list comprehensions, lists, modules, pattern matching, polymorphism, property-based testing, reasoning about functional programs, recursive functions, sets, strings, tail recursion, trees, tupling, type checking, type inference, types, types and type classes, unification, user-defined types

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Overview

- Introduction to the λ-Calculus
- Encoding Data Types

Introduction to the λ-Calculus

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- that is, Turing-complete is the same as definable in the λ-Calculus
- λ-Calculus is underlying much of functional programming

Syntax – λ-Terms

- grammar

$$\begin{array}{rcll} t & ::= & x & \text{variable/atom} \\ & | & (t\ t) & \text{application} \\ & | & (\lambda x.\, t) & \text{(lambda) abstraction} \end{array}$$

- set of lambda terms is denoted Λ

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Examples

$$(\lambda x.\, y)$$

$$(\lambda x.\, (\lambda y.\, x))$$

$$(\lambda x.\, (\lambda y.\, (\lambda z.\, ((x\ z)\ (y\ z)))))$$

$$(\lambda x.\, ((\lambda y.\, (\lambda z.\, (z\ y)))\ x))$$

Conventions (to ease reading and writing)

- outermost parentheses are dropped
- application binds stronger than abstraction, e.g., $\lambda x.\, x\ z$ is equal to $\lambda x.\, (x\ z)$ and not to $(\lambda x.\, x)\ z$
- applications associate to the left
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Note

- nested lambdas are "functions with multiple arguments"
- e.g., $\lambda xyz.\, t$ is a function taking 3 arguments

λ-Terms and Haskell

Haskell

- `(\x -> x + 1)`
- `(\x -> x + 1) 2` $= 3$
- `if True then 1 else 0` $= 1$
- `(,) 2 4` $=$ `(2, 4)`
- `fst (2, 4)` $= 2$

λ-Calculus

- $\lambda x.\,\text{add}\ x\ 1$
- $(\lambda x.\,\text{add}\ x\ 1)\ 2$
- if True $1\ 0$
- Pair $2\ 4$
- fst $(\text{Pair}\ 2\ 4)$

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`(\x -> x + 1) 2 = 3`	$(\lambda x.\,\text{add}\ x\ 1)\ 2$
`if True then 1 else 0 = 1`	if True $1\ 0$
`(,) 2 4 = (2, 4)`	Pair $2\ 4$
`fst (2, 4) = 2`	fst $(\text{Pair}\ 2\ 4)$

Remark

- '0', '1', '2', '4', 'add', 'fst', 'if', 'Pair', and 'True' are abbreviations for more complex λ-terms
- supposed to "encode" the behavior of 0, 1, 2, 4, (+), ...

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The β-Rule (informal definition)

- intuition: apply "function" to "argument"
- in λ-Calculus both "functions" and "arguments" are λ-terms
- the β-rule

$$(\lambda x.\, s)\; t \quad \to_\beta \quad s[x := t]$$

- in words: *when applying function $(\lambda x.\, s)$ to input t, replace every occurrence of x in body of function (that is, s) by t*

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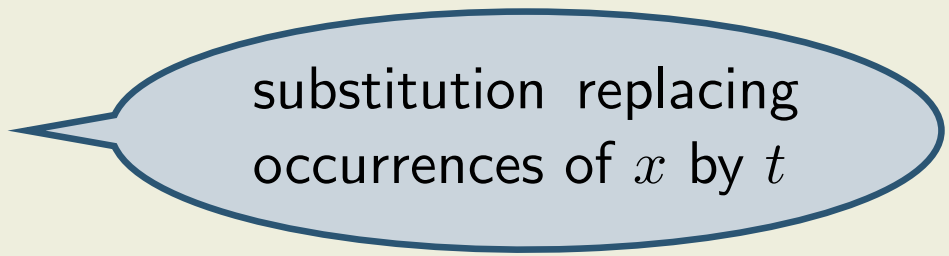

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$$\lambda x.\, x \qquad \text{no } \beta\text{-step possible}$$

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Solution

- do not allow arbitrary substitution (freshness constraints)
- work modulo consistent renaming of bound variables (α-equivalence)

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Example

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Free Variables and Freshness

- set of free variables of a term

$$\mathcal{F}(x) = \{x\}$$
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Free Variables and Freshness

- set of free variables of a term

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- α-equivalence class of $\lambda x.\, y\ x$ (remember: all elements considered the same term)

$$\{\lambda x.\, y\ x, \quad \lambda z.\, y\ z, \quad \lambda a.\, y\ a, \quad \lambda x'.\, y\ x', \quad \lambda b_1.\, y\ b_1, \quad \ldots\}$$

Substitutions

- substitution (for terms) is function from variables to terms
- we only need substitutions replacing a single variable
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Example

- consider $\sigma = [x := \lambda x.\, x]$
- then $\sigma(x) = \lambda x.\, x$ and
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- inductive definition of one step β-reduction

$$\frac{}{(\lambda x.\, t)\ u \to_\beta t[x := u]}\ \text{(root)} \qquad \frac{s \to_\beta t}{s\ u \to_\beta t\ u}\ \text{(app-l)}$$

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- a nonempty sequence (that is, $n > 0$) is denoted by $s \to_\beta^+ t$

Exercise

- consider $\Omega = (\lambda x.\, x\ x)\ (\lambda x.\, x\ x)$,
- $K = \lambda xy.\, x$,
- $K_* = \lambda xy.\, y$, and
- $I = \lambda x.\, x$
- reduce the following λ-terms

$$K\ \Omega$$
$$K_*\ \Omega$$
$$I\ \Omega$$

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Examples

- $\lambda x.\, x$ is in NF
- $(\lambda x.\, x)\ y$ is not in NF, since $(\lambda x.\, x)\ y \to_\beta y$ (with NF y)

Encoding Data Types

Booleans and Conditionals

Haskell	λ-Calculus	
• `True`	• True $= \lambda xy.\, x$	"ignore second argument"
• `False`	• False $= \lambda xy.\, y$	"ignore first argument"
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Examples

if True $x\ y$

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Natural Numbers – Church Numerals

- define n-fold application of "function"

$$s^0\ t = t$$
$$s^{n+1}\ t = s\ (s^n\ t)$$

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function that applies first argument n-times to second argument

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Haskell vs. λ-Calculus

Haskell	λ-Calculus
`0`	$0 = \lambda f x.\ x$
`1`	$1 = \lambda f x.\ f\ x$
`n`	$n = \lambda f x.\ f^n\ x$
`(+)`	$\text{add} = \lambda m n f x.\ m\ f\ (n\ f\ x)$
`(*)`	$\text{mul} = \lambda m n f.\ m\ (n\ f)$
`(^)`	$\text{exp} = \lambda m n.\ n\ m$

Pairs

Haskell	λ-Calculus
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• `fst`	• fst $= \lambda p.\, p$ True
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Lists

Haskell	λ-Calculus
`(:)`	Cons $= \lambda xy.$ Pair False (Pair $x\ y$)
`head`	head $= \lambda z.$ fst (snd z)
`tail`	tail $= \lambda z.$ snd (snd z)
`[]`	Nil $= \lambda x.\, x$
`null`	null $=$ fst

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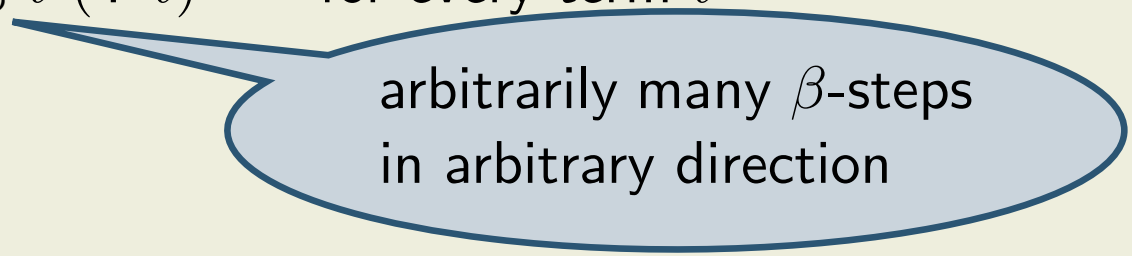

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Example – Length

- recall that length $=$ Y g with
  $g = \lambda f x.\, \mathsf{if}\ (\mathsf{null}\ x)\ 0\ (\mathsf{add}\ 1\ (f\ (\mathsf{tail}\ x)))$
- by fixed point property we obtain length $\equiv_\beta g$ length, taking care of replacing additional parameter f in g by definition of length

Exercises (for November 17th)

1. Read the lecture notes on the lambda calculus until Section 5.
2. Use the conventions to simplify $\lambda x.\,(\lambda y.\,(\lambda z.\,((z\ (x\ y))\ x)))$ and drop the conventions in the term $\lambda abcd.\,a\ b\ c\ d$.
3. Consider F $= \lambda f x y.\,f\ y\ x$. What does F do? What does F $(\lambda x y.\,x)$ do? Reduce F $(\lambda x y.\,x)$ to NF.
4. Using `(\\)` and `union` from `Data.List` as well as the types
   ```
   type Id = String
   data Term = Var Id | App Term Term | Abs Id Term
   ```
   implement functions `vars` and `fvs` (both of type `Term -> [Id]`) computing the set (in the form of a list without duplicates) of all variables and free variables occurring in a term, respectively.
5. Implement a function `freshName` that, given a variable `x` and a list of variables `ys`, computes a pair `(x', f)` where `x'` is fresh for `ys` and `f :: Term -> Term` replaces all occurrences of `x` in a term by `x'`.
6. With the help of `vars`, `fvs`, and `freshName`, implement a function `subst :: String -> Term -> Term -> Term`, where `subst` $x\ s\ t$ computes $t[x := s]$, satisfying the freshness constraint.