Functional Programming

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Lecture 7

Topics

abstract data types, algebraic data types, binary search trees, combinator parsing, efficiency, encoding data types as lambda-terms, evaluation strategies, formal verification, first steps, guarded recursion, Haskell introduction, higher-order functions, historical overview, implementing a type checker, induction, infinite data structures, input and output, lambda-calculus, lazy evaluation, list comprehensions, lists, modules, pattern matching, polymorphism, property-based testing, reasoning about functional programs, recursive functions, sets, strings, tail recursion, trees, tupling, type checking, type inference, types, types and type classes, unification, user-defined types

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Overview

- Mathematical Induction
- Induction over Lists
- Structural Induction
- Formal Verification of Functional Programs

Mathematical Induction

When to use Mathematical Induction?

- prove that some property P holds for all natural numbers
- more formally, prove:

$$\forall n.\, P(n) \qquad \text{(where } n \in \mathbb{N}\text{)}$$

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- mathematical induction consists of two steps:
- first prove base case

$$P(0)$$

- then step case

$$\forall k.\, (P(k) \longrightarrow P(k+1))$$

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assume $P(k)$ (induction hypothesis), show $P(k+1)$

Why does this Work?

- we have two ingredients:
  1. P is true for 0
  2. if P is true for arbitrary k it is also true for $k + 1$
- and want to show P for every natural number ($\forall n. P(n)$)

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Idea

- we can reach arbitrary n
- such that $P(n)$
- hence, $\forall n.\, P(n)$

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$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &\stackrel{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

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- base case may be changed
- e.g., if base case $P(1)$, property holds for all $n \geq 1$

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- first domino will fall
- if a domino falls also its right neighbor falls

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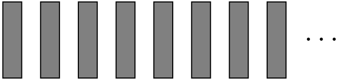

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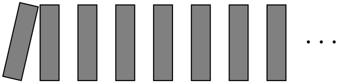

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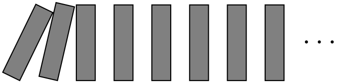

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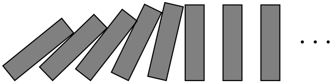

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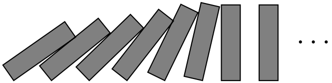

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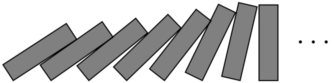

Induction over Lists

Algebraic Data Type of Lists

```
data [a] = [] | (:) a [a]
```

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Notes

- lists are recursive structures
- non-recursive constructor (base case): `[]`
- recursive constructor (step case): x : xs

Induction Principle for Lists – Informally

- to show $P(xs)$ for all lists xs
- show base case: $P(\texttt{[]})$
- show step case: $P(xs) \longrightarrow P(x \texttt{ : } xs)$ for arbitrary x and xs

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Remark

- for lists, P can be seen as function `p :: [a] -> Bool`

Exercise – Nil is right identity of append

- definition of append

```
   []    ++ ys  =  ys
(x:xs) ++ ys  =  x : (xs ++ ys)
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- definition of append

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   []    ++ ys  =  ys
(x:xs)   ++ ys  =  x : (xs ++ ys)
```

- prove that [] is right identity of ++, that is,

$$xs \mathbin{+\!\!+} [] = xs$$

Exercise – Associativity of append

- recall

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Exercise – Associativity of append

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- prove that ++ is associative, that is,

$$(xs +\!\!+\ ys) +\!\!+\ zs = xs +\!\!+\ (ys +\!\!+\ zs)$$

Exercise – Length and append

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- definition

```
length []      =  0
length (_:xs)  =  1 + length xs
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- prove that length of combined list is sum of lengths, that is,

$$\texttt{length}\ (xs \mathbin{+\!+} ys) = \texttt{length}\ xs + \texttt{length}\ ys$$

Structural Induction

Example – Terms

```
type Id = String
data Term = Var Id
          | App Term Term
          | Abs Id Term
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General Structures – Induction Principle

- for every non-recursive constructor, show base case
  - base case: $P(\texttt{Var x})$
- for every recursive constructor, show step case
  - step case 1: $(P(\texttt{s}) \wedge P(\texttt{t})) \longrightarrow P(\texttt{App s t})$
  - step case 2: $P(\texttt{t}) \longrightarrow P(\texttt{Abs x t})$

Example – Binary Trees

```
data BTree a = Empty
             | Node a (BTree a) (BTree a)
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Induction Principle for Binary Trees

$$(P(\texttt{Empty}) \land \forall x. \forall l. \forall r. ((P(l) \land P(r)) \longrightarrow P(\texttt{Node}\ x\ l\ r))) \longrightarrow \forall t. P(t)$$

Exercise – Perfect Binary Trees

- a binary tree is perfect if all leaf nodes have same depth

```
perfect Empty         = True
perfect (Node _ l r) =
  height l == height r && perfect l && perfect r

height Empty         = 0
height (Node _ l r) =
  max (height l) (height r) + 1

size Empty         = 0
size (Node _ l r) = size l + size r + 1
```

- lemma: a perfect binary tree t of height n has exactly $2^n - 1$ nodes, that is,

$$P(t) = \left(\texttt{perfect}\ t \longrightarrow \texttt{size}\ t = 2^{\texttt{height}\ t} - 1\right)$$

Formal Verification of Functional Programs

Isabelle/HOL in a Nutshell

Obvious question:

- What is Isabelle?

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Common answer:

- An LCF-style proof assistant.

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Typical follow-up questions:

- What is a proof assistant?
- What does LCF-style mean?
- ...

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- combination of automated theorem prover and proof checker

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Example

- automatic methods: classical tableaux (`blast`), equational reasoning (`simp`), combination of former (`auto`), ...
- manual steps: induction (`induct`), case analysis (`cases`), ...

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Example

- functions `assume : cterm -> thm` and `implies_elim : thm -> thm -> thm`
- implement inference rules

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \Longrightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

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Example

certified term

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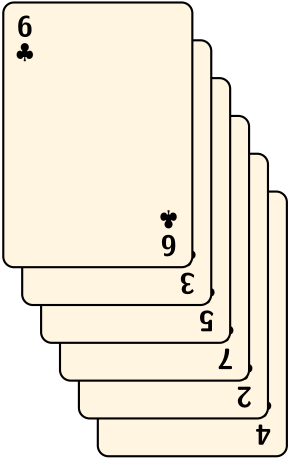
Higher-Order Logic

- HOL = Functional Programming + Logic
- data types (`datatype`)
- recursive functions (`fun`)
- logical operators ($\wedge$, $\vee$, $\longrightarrow$, $\forall$, $\exists$, ...)

Example – Insertion Sort



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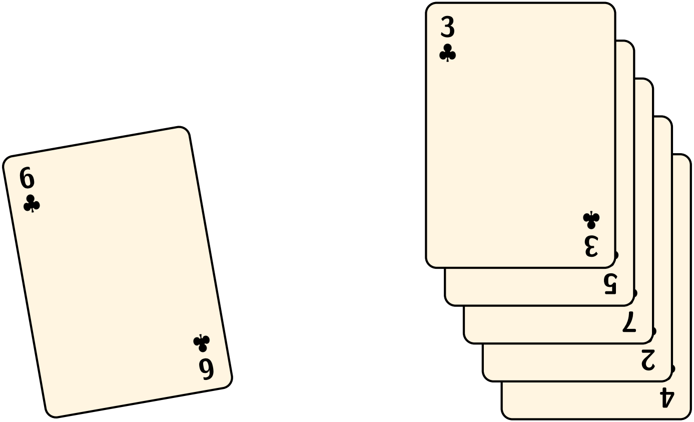

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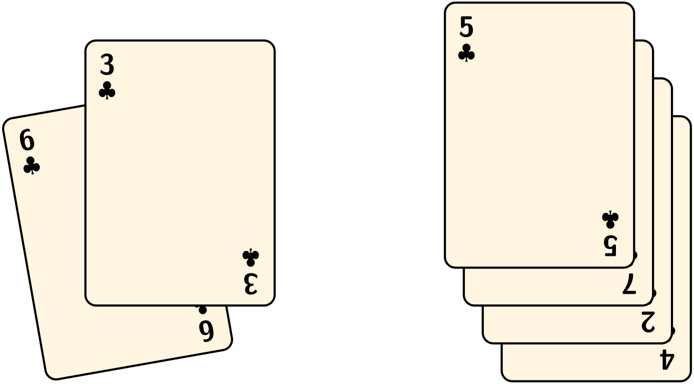

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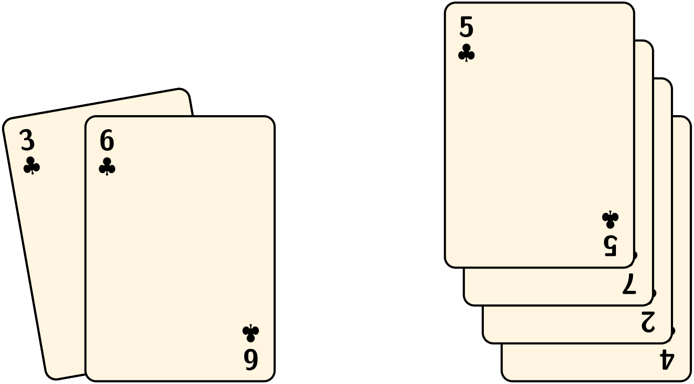

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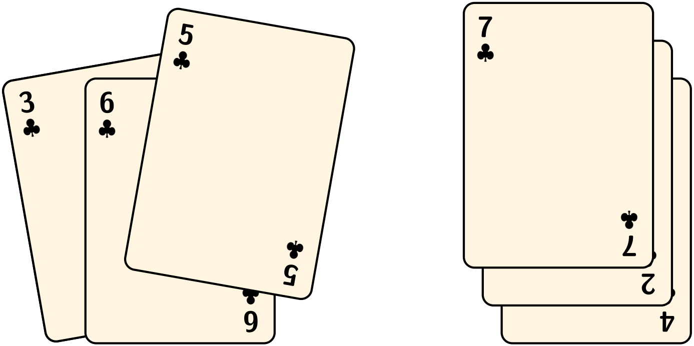

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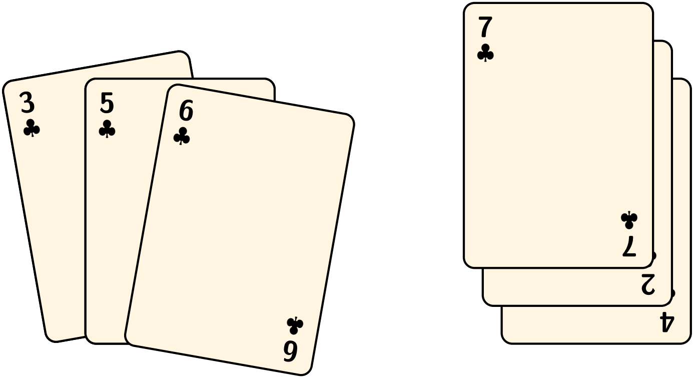

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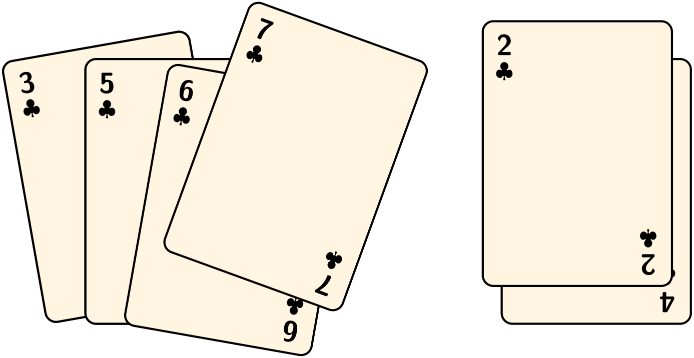

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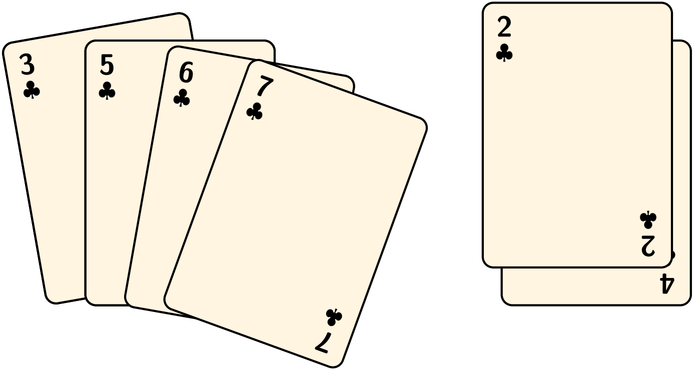

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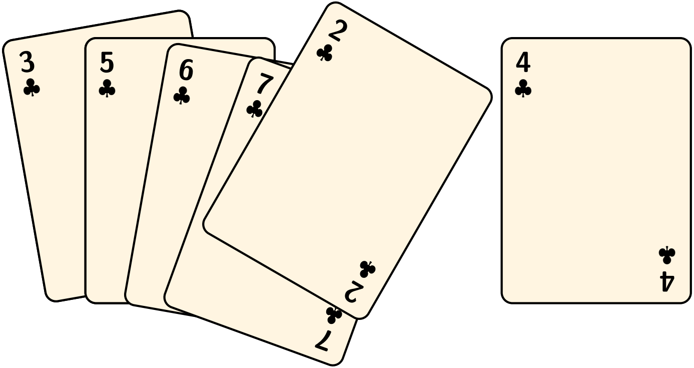

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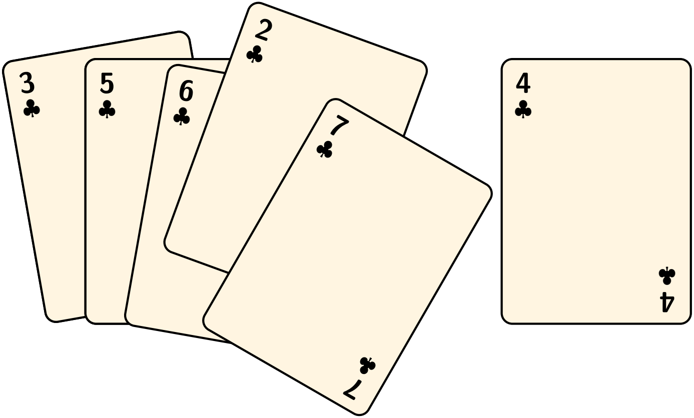

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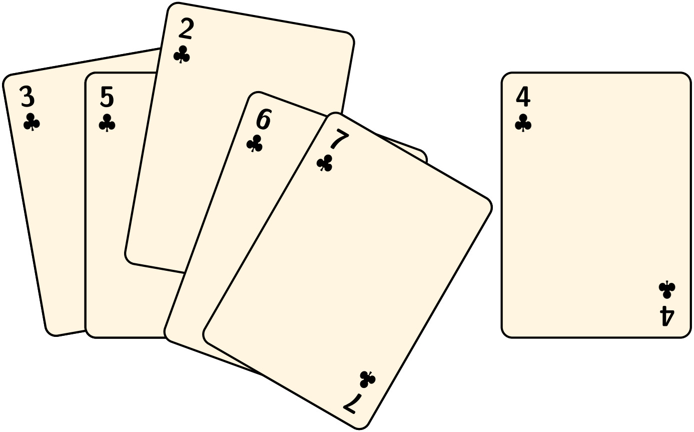

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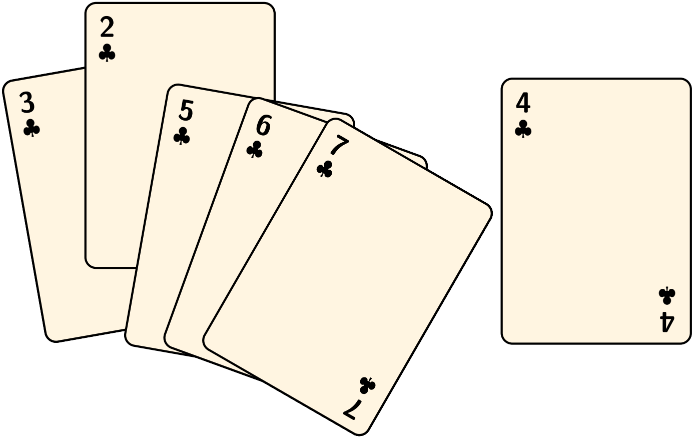

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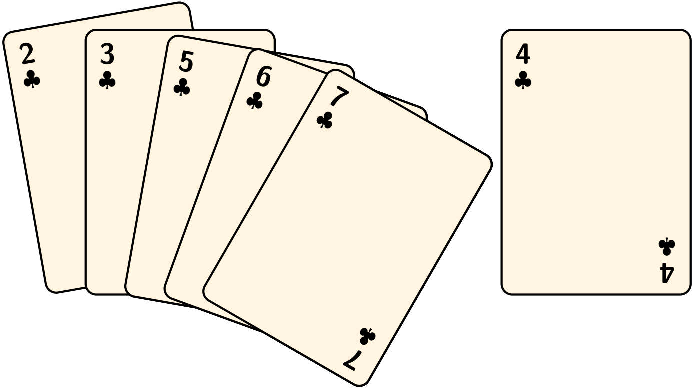

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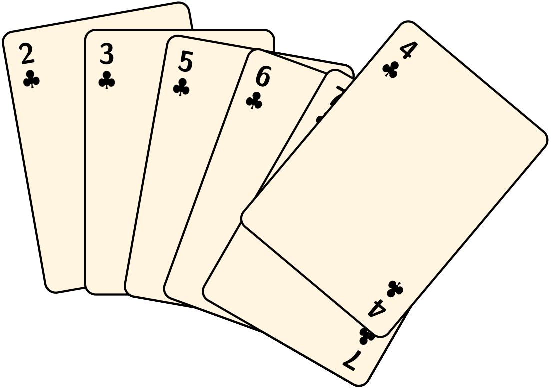

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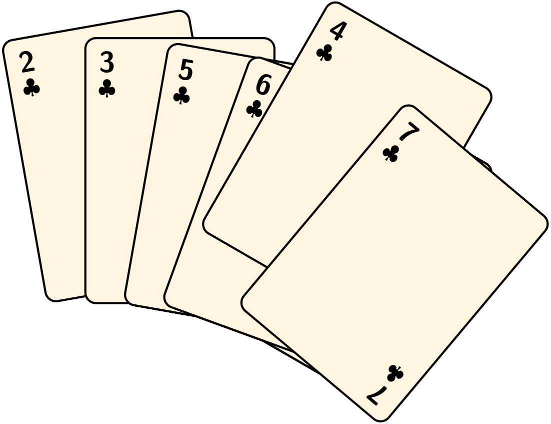

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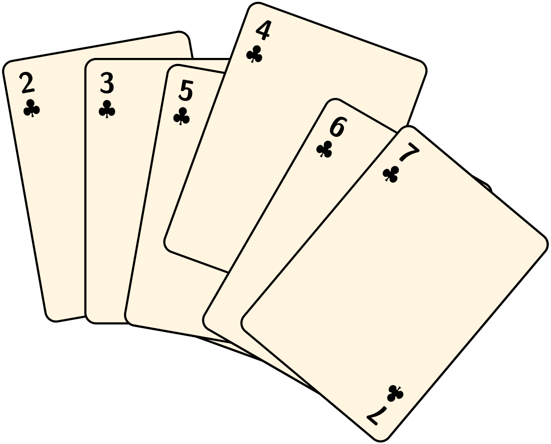

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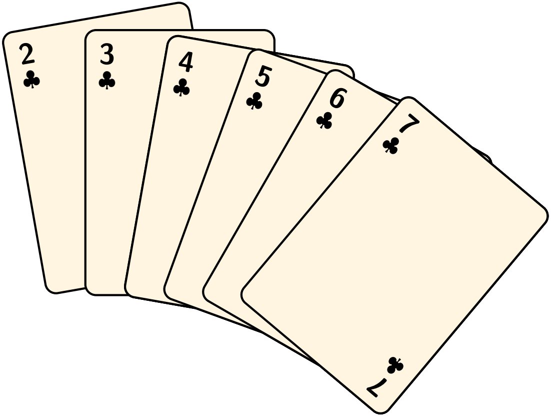

A Functional Implementation

- inserting an element into a sorted list

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insert x [] = [x]
insert x (y:ys) =
  if x <= y then x : y : ys
  else y : insert x ys
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- see `Insertion_Sort.thy`

Exercises (for December 1st)

1. Read the lecture notes on reasoning about functional programs.
2. Download Isabelle/HOL from `http://isabelle.in.tum.de` and start its Isabelle/jEdit interface.
3. Prove the following equation by induction:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

4. Prove $\texttt{rev}\ (xs \mathbin{+\!+} ys) = \texttt{rev}\ ys \mathbin{+\!+} \texttt{rev}\ xs$ for

```
rev []       =  []
rev (x:xs)   =  rev xs ++ [x]
```

using the equations

$$xs \mathbin{+\!+} [] = xs \qquad (\star)$$

$$(xs \mathbin{+\!+} ys) \mathbin{+\!+} zs = xs \mathbin{+\!+} (ys \mathbin{+\!+} zs) \qquad (\star\star)$$

5. Prove Exercise 4 in Isabelle/HOL using the HOL data type

```
datatype 'a list = Nil | Cons 'a "'a list"
```

and your own definitions of reversal (`rev`) and append (`app`).

6. Prove $\texttt{rev}\ (\texttt{rev}\ xs) = xs$ in Isabelle/HOL.