A Formally Verified Solver for Homogeneous Linear Diophantine Equations*

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Master Seminar 1

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Quiz – How many unifiers?

$$\mathsf{f}(x, y) \approx \mathsf{f}(z, z)$$

#unifiers: ?

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$$\mathsf{f}(x, y) \;\approx\; \mathsf{f}(z, z)$$

#unifiers: 1

$$\{x \mapsto z, y \mapsto z\}$$

Quiz – How many AC unifiers?

$$x \cdot y \approx z \cdot z$$

#unifiers: ?

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$$x \cdot y \approx z \cdot z$$

#unifiers: 5

minimal complete set of AC unifiers:

$$\{x \mapsto z_3, \quad y \mapsto z_3, \quad z \mapsto z_3\}$$
$$\{x \mapsto z_1 \cdot z_1, \quad y \mapsto z_2 \cdot z_2, \quad z \mapsto z_1 \cdot z_2\}$$
$$\{x \mapsto z_1 \cdot z_1 \cdot z_3, \quad y \mapsto z_3, \quad z \mapsto z_1 \cdot z_3\}$$
$$\{x \mapsto z_3, \quad y \mapsto z_2 \cdot z_2 \cdot z_3, \quad z \mapsto z_2 \cdot z_3\}$$
$$\{x \mapsto z_1 \cdot z_1 \cdot z_3, \quad y \mapsto z_2 \cdot z_2 \cdot z_3, \quad z \mapsto z_1 \cdot z_2 \cdot z_3\}$$

Quiz – How many AC unifiers?

$$x \cdot y \ \approx \ z \cdot z \cdot z$$

#unifiers: ?

Quiz – How many AC unifiers?

$$x \cdot y \approx z \cdot z \cdot z$$

#unifiers: 13

Quiz – How many AC unifiers?

$$v \cdot x \cdot y \;\approx\; z \cdot z \cdot z$$

#unifiers: ?

Quiz – How many AC unifiers?

$$v \cdot x \cdot y \;\approx\; z \cdot z \cdot z$$

#unifiers: 981

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Homogeneous Linear Diophantine Equations (HLDEs)

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m \quad = \quad b_1 y_1 + b_2 y_2 + \cdots + b_n y_n$$

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left-hand side coefficients

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Example

$$x_1 + x_2 = 2 y_1$$

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$$[1, 1] \bullet [x_1, x_2] = [2] \bullet [y_1]$$

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Remark

we represent HLDEs by lists of coefficients, e.g., `([1,1],[2])`

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- given $x_1 + x_2 = 2 y_1$, represented by [1,1] and [2]
- consider potential solutions

x_1 x_2 y_1

[0,0], [0] ?

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x1 x2    y1
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x_1 x_2 y_1

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[1,1], [1] ✔
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[3,1], [2] ?

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[1,1],  [1] ✔
    ⋮
[2,0],  [1] ✔
    ⋮
[3,1],  [2] ✔ (but not minimal)
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Bounding Minimal Solutions

Lemma (Huet)
if $(x, y) \in \mathcal{M}(a, b)$ then $x_i \leq \max(b)$ and $y_j \leq \max(a)$

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Example

- for $a = $ `[1,1]` and $b = $ `[2]`
- 18 potential solutions $(3^2 \cdot 2^1)$

```
[([0,0],[0]), ([1,0],[0]), ([2,0],[0]), ([0,1],[0]),
 ([1,1],[0]), ([2,1],[0]), ([0,2],[0]), ([1,2],[0]),
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```

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```

- containing 4 actual solutions $(a \bullet x = b \bullet y)$

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- containing 4 actual solutions $(a \bullet x = b \bullet y)$

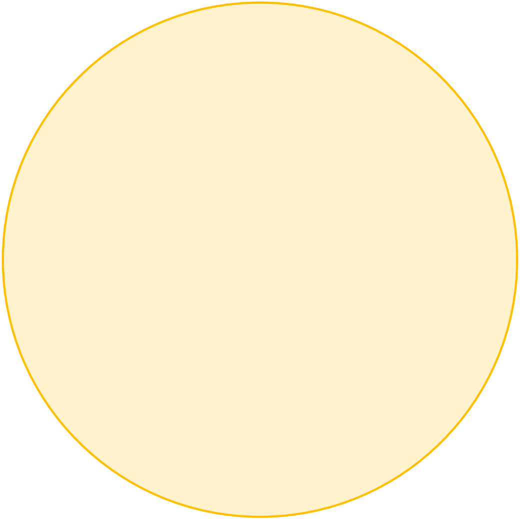
```
[([0,0],[0]),([2,0],[1]),([1,1],[1]),([0,2],[1])]
```

- of which 3 are minimal (w.r.t. $<_{\mathsf{v}}$)

```
[([2,0],[1]), ([0,2],[1]), ([1,1],[1])]
```

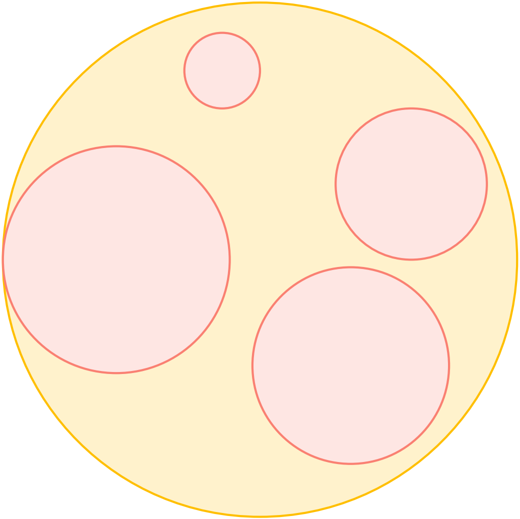
Computing Minimal Complete Sets of Solutions

1. generate potential solutions (crude overapproximation)



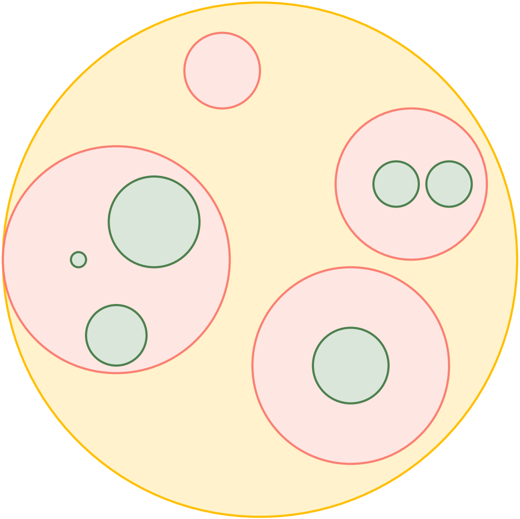
Computing Minimal Complete Sets of Solutions

1. generate potential solutions (crude overapproximation)
2. check for actual solutions



Computing Minimal Complete Sets of Solutions

1. generate potential solutions (crude overapproximation)
2. check for actual solutions
3. minimize remaining set of candidates



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- given bound `b` and coefficients `cs`

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- compute all vectors of length equal to `length cs` within bound

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gen b [] = [[]]
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- given bounds `a`, `b` and coefficients `as`, `bs`

Example

- equation $x_1 + x_2 = 2y_1$, `a` $=$ `2`, `b` $=$ `1`, `as` $=$ `[1,1]`, `bs` $=$ `[2]`

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[0,0],    [1,0],    [2,0],    [0,1],    [1,1],
[2,1],    [0,2],    [1,2],    [2,2],
```

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- (solutions are generated in reverse lexicographic order $<_{\mathsf{rlex}}$)

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```
[0,0], [0]   [1,0], [0]   [2,0], [0]   [0,1], [0]   [1,1], [0]
[2,1], [0]   [0,2], [0]   [1,2], [0]   [2,2], [0]
[0,0], [1]   [1,0], [1]   [2,0], [1]   [0,1], [1]   [1,1], [1]
[2,1], [1]   [0,2], [1]   [1,2], [1]   [2,2], [1]
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- (solutions are generated in reverse lexicographic order $<_{\text{rlex}}$)

$x <_{\text{rlex}} y$ iff $\exists i.\, x_i < y_i \wedge \forall j > i.\, x_j = y_j$

Example

- equation $x_1 + x_2 = 2y_1$, a = 2, b = 1, as = [1,1], bs = [2]

```
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[2,1], [1]  [0,2], [1]   [1,2], [1]   [2,2], [1]
```

Phase 2 – Check

- drop non-solutions

```
check as bs = filter (isSolution as bs)
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Example

- equation $x_1 + x_2 = 2\,y_1$, a = 2, b = 1, as = [1,1], bs = [2]

```
              [1,0], [0]   [2,0], [0]   [0,1], [0]   [1,1], [0]
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[0,0], [1]    [1,0], [1]   [2,0], [1]   [0,1], [1]   [1,1], [1]
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[0,2], [1]

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check as bs = filter (isSolution as bs)
```

Phase 3 – Minimize

-
```
minimize [] = []
minimize ((x,y):xs) =
    (x,y) : filter (x ++ y ≮v ) (minimize xs)
```

Example

- equation $x_1 + x_2 = 2y_1$, a $=$ 2, b $=$ 1, as $=$ [1,1], bs $=$ [2]

[2,0], [1]   [1,1], [1]

[0,2], [1]

Phase 2 – Check

- drop non-solutions

```
check as bs = filter (isSolution as bs)
```

Phase 3 – Minimize

- ```
  minimize [] = []
  minimize ((x,y):xs) =
      (x,y) : filter (x ++ y ≮v ) (minimize xs)
  ```

Remark

if $x <_{\mathsf{v}} y$ then $x <_{\mathsf{rlex}} y$

Example

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[2,0], [1] [1,1], [1]

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solutions as bs =
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[1,1]	[2]	3	0.001

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Special Solutions

- given i and j, unique special solution is

$$0 \cdots \mathsf{lcm}(a_i, b_j)/a_i \cdots 0, 0 \cdots \mathsf{lcm}(a_i, b_j)/b_j \cdots 0$$

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Example

- equation $x_1 + x_2 = 2y_1$
- special solutions

```
specialSolutions [1,1] [2] = [([2,0],[1]),([0,2],[1])]
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- only 1 non-zero x_i and y_j
- all special solutions are minimal
- it remains to compute non-special solutions (that is, those minimal solutions that are not special)

Example

- equation $x_1 + x_2 = 2\,y_1$
- special solutions

```
specialSolutions [1,1] [2] = [([2,0],[1]),([0,2],[1])]
```

Non-Special Solutions

Lemma (Huet)

if (x, y) is non-special minimal solution then

- $y_j \leq \texttt{maxy}\ x\ j$
- $\texttt{take}\ a\ k \bullet \texttt{take}\ x\ k \leq b \bullet y$
- $\texttt{take}\ b\ l \bullet \texttt{take}\ y\ l \leq a \bullet \texttt{map}\ (\texttt{maxx}\ (\texttt{take}\ y\ l))\ [0..m-1]$

where

$$\texttt{maxx}\ y\ i = \textbf{if}\ i < m \wedge D_i\ y \neq 0\ \textbf{then}\ \mathsf{min}(D_i\ y)\ \textbf{else}\ \mathsf{max}(b)$$

$$\texttt{maxy}\ x\ j = \textbf{if}\ j < n \wedge E_j\ x \neq 0\ \textbf{then}\ \mathsf{min}(E_j\ x)\ \textbf{else}\ \mathsf{max}(a)$$

$$D_i\ y = \{\mathsf{lcm}(a_i, b_j)/a_i - 1 \mid j < |y| \wedge y_j \geq \mathsf{lcm}(a_i, b_j)/b_j\}$$

$$E_j\ x = \{\mathsf{lcm}(a_i, b_j)/b_j - 1 \mid i < |x| \wedge x_i \geq \mathsf{lcm}(a_i, b_j)/a_i\}$$

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Improved Bounds on Minimal Solutions

(Clausen and Fortenbacher)

if (x, y) is minimal solution then $x_i \leq \max^{\neq 0}\ y\ b$ and $y_j \leq \max^{\neq 0}\ x\ a$

Merging Generate and Check

- compute all vectors of length equal to `length cs` whose elements and "partial sums" satisfy `p`

```
incs p c i (xs,s) =
    if p (i:xs) t then (i:xs,t) : incs p c (i+1) (xs,s)
    else []
  where
    t = s + c*i

genCheck p [] = [([],0)]
genCheck p (c:cs) =
  concat (map (incs p c 0) (genCheck p cs))
```

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genCheck p [] = [([],0)]
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```

- compute potential solutions within bounds

```
generateCheck as bs =
    tail [(x, y) | (y, _) <- genCheck q bs,
                   (x, _) <- genCheck (p y) as]
  where
    p ys (x:_) s = s <= bs `dp` ys && x <= maxne0 ys bs
    ...
```

An Improved Algorithm

- additional check phase

```
check' as bs = filter (\(xs, ys) ->
  all (<= maxne0 xs as) ys &&
  isSolution as bs xs ys &&
  all (\j -> ys !! j <= maxy xs j) [0..length bs - 1])
```

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- computing non-special solutions

```
nonSpecialSolutions as bs =
    minimize (check' as bs (generateCheck as bs))
```

An Improved Algorithm

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nonSpecialSolutions as bs =
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```
solutions' as bs =
  specialSolutions as bs ++ nonSpecialSolutions as bs
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- the algorithm

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solutions' as bs =
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```

Lemma

`solutions'` and `solutions` compute the same results

Example

```
generateCheck [1,1] [2]
```

```
nonSpecialSolutions [1,1] [2]
```

```
solutions' [1,1] [2]
```

Example

```
generateCheck [1,1] [2]
```

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nonSpecialSolutions [1,1] [2]
```

```
solutions' [1,1] [2]
 = specialSolutions [1,1] [2] ++ nonSpecialSolutions ...
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Example

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generateCheck [1,1] [2]
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Example

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generateCheck [1,1] [2]
 = tail [(x,y) | y <- genCheck q [2],
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Example

```
generateCheck [1,1] [2]
 = tail [(x,y) | y <- genCheck q [2],
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 = tail ([(x,[0]) | x <- genCheck (p [0]) [1,1]] ++ [(x,[1])
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nonSpecialSolutions [1,1] [2]
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 = tail ([(x,[0]) | x <- [[0,0]]] ++ [(x,[1]) | ...])
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 = [([0,0],[1]),([1,0],[1]),([2,0],[1]),([0,1],[1]),
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[1,1,1,2,3]	[1,1,2,2]	44	0.01
[2,5,9]	[1,2,3,7,8]	119	8.6
[2,2,2,3,3,3]	[2,2,2,3,3,3]	138	0.06
[1,2,2,5,9]	[1,2,3,7,8]	345	517.4
[1,4,4,8,12]	[3,6,9,12,20]	232	67.4

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- and of simple solver computing minimal complete sets of solutions
- clear separation of 3 phases: generate, check, and minimize
- which greatly simplifies proofs
- basis for computing minimal complete sets of AC unifiers
- improved efficiency by partially merging generate and check phases