(Formalization) of Social Choice Theory

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Department of Computer Science

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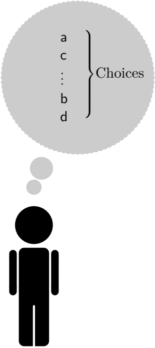
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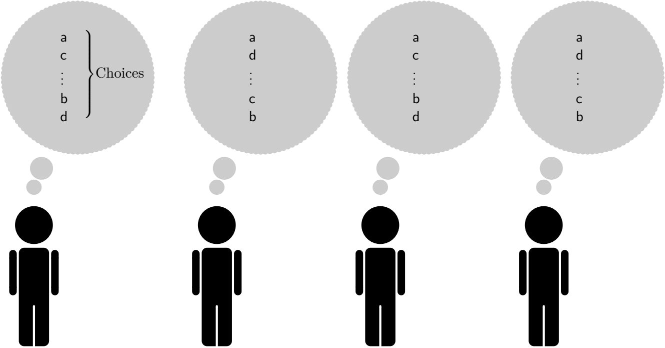
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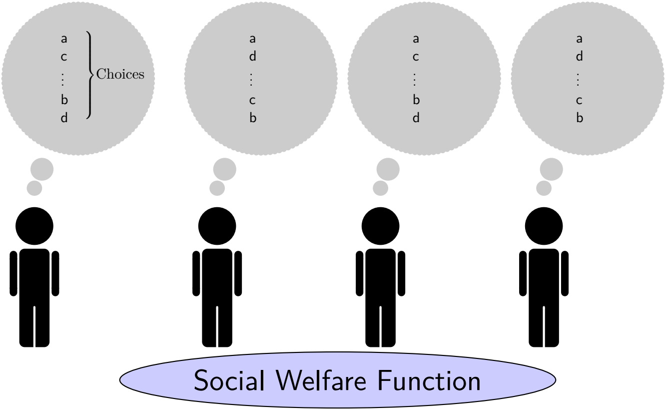
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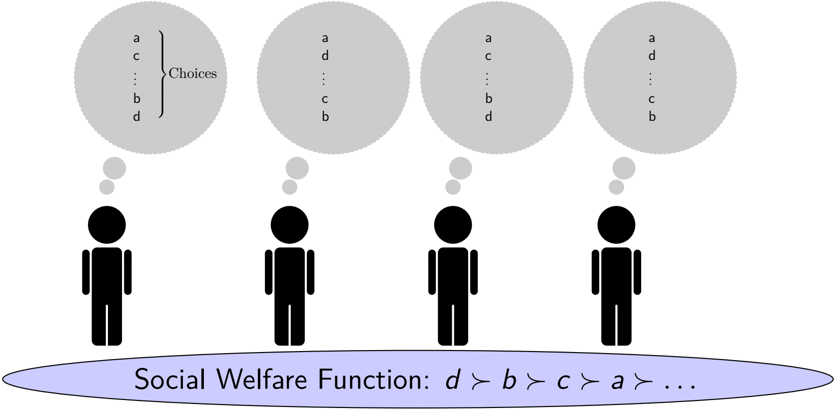
Social Welfare Function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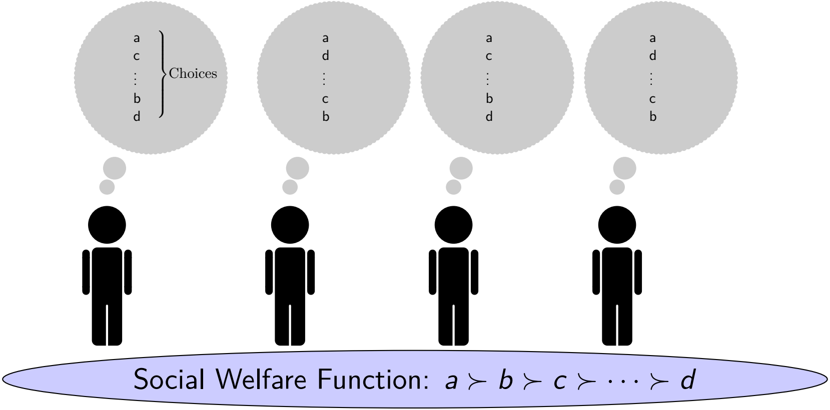
Social Welfare Function: $d \succ b \succ c \succ a \succ \ldots$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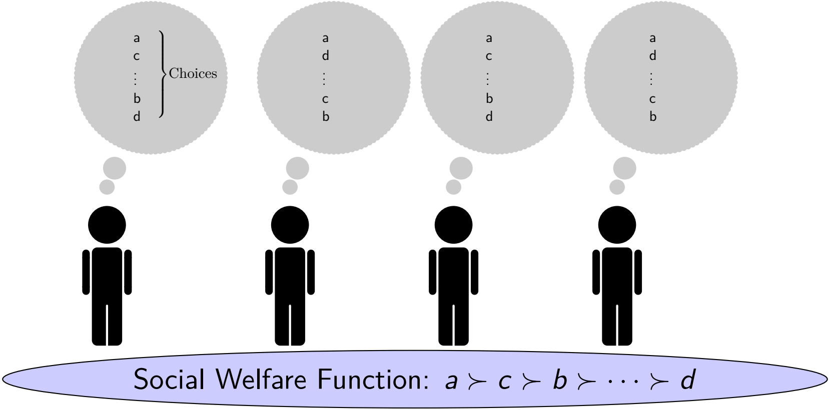
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Social Welfare Function: $a \succ c \succ b \succ \cdots \succ d$

Definition (Preference Relation over $\mathcal{C}$)

binary relation, $\succeq$ over $\mathcal{C}$
transitive, reflexive

Definition (Rational Preference Relation over $\mathcal{C}$)

binary relation, $\succeq$ over $\mathcal{C}$
transitive, reflexive, total

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$x \succ y \equiv x \succeq y \land \neg y \succeq x$

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$x \succ y \equiv x \succeq y \land \neg y \succeq x$

Example (Preference Relation over a, b, c for Alice)

$$P_{Alice} := \{(b, c), (b, a), (c, a)\} \equiv \begin{array}{c} P_{Alice} \\ \hline \text{b} \\ \text{c} \\ \text{a} \end{array} \equiv b \succ_{Alice} c \succ_{Alice} a$$

Social Welfare Functions

The Setup

- A non empty set of n Individuals I
- A set of choices, $\mathcal{C}$ with $|\mathcal{C}| \geq 3$
- $\mathcal{P}$ is the set of all possible preference relations over $\mathcal{C}$

Social Welfare Functions

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Notation

$x \succeq_{P_i} y \equiv (x, y) \in P_i$
$x \succeq_i y \equiv x \succeq_{P_i} y$
$x \succ_i y \equiv x \succeq_i y \wedge \neg y \succeq_i x$
$P_i \in \mathcal{P}$ is the set of pairs describing preferences of agent i

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Assumption

Agents preferences are strict: $\forall i \in I.\ x \succeq_i y \rightarrow x \succ_i y$

Social Welfare Function

Definition (Social Welfare Function (SWF))

A social welfare function is a mapping $\mathcal{F} : \mathcal{P}^n \to \mathcal{P}$

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Notation

$x \succeq y \equiv x \succeq_{SWF(P_1,\dots,P_n)} y \equiv (x, y) \in \mathcal{F}(P_1, \dots, P_n)$
$x \succ y \equiv x \succeq y \wedge \neg y \succeq x$
Preference Profile P where $P := (P_1, \dots, P_n) \in \mathcal{P}^n$

Definition (Dictatorial SWF)

- A Social Welfare Function $\mathcal{F}$ is dictatorial if there exists individual $i \in I$ such that $\forall xy \in \mathcal{C}.\ x \succ_i y \rightarrow x \succ y$
- $\mathcal{F}$ is dictatorial for i if $\mathcal{F}$ is a projection π_i^n.

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1	...	i	...	n	$\mathcal{F}$
...	...	b	⋮	...	b
⋮	⋮	X	⋮	⋮	X
...	...	a	...	...	a
⋮	⋮	Y	⋮	⋮	Y
...	...	c	...	...	c
⋮	...	Z	...	...	Z

Pivotal Individual

Definition (Decisive/Pivotal)

- An agent i is pivotal for $a, b \in \mathcal{C}$ at $P_i, \dots P_n$ if by changing the relation between a and b the SWF changes this relation as well.
- $a \succ_i b \iff a \succ_{SWF(P_1,\dots,P_i,\dots,P_n)} b$

Unanimity

Unanimity

- If all individuals have the same relation between two alternatives a and b, the SWF has to reflect that relation.
- $\forall a\ b \in \mathcal{C}.\ (\forall i \in I.\ a \succeq_i b) \rightarrow a \succeq b$

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- $\forall a\ b \in \mathcal{C}.\ (\forall i \in I.\ a \succeq_i b) \rightarrow a \succeq b$

1	...	...	...	n	$\mathcal{F}$
a	...	...	...	a	⋮
⋮	a	⋮	⋮	⋮	a
...	b	...	a	...	⋮
⋮	⋮	⋮	⋮	⋮	⋮
b	...	a	b	...	b
⋮	...	b	...	b	⋮

Independence of Irrelevant Alternatives

Independence of Irrelevant Alternatives (IIA)

- Social preference between two alternatives a and b only depends on the voters' preferences between a and b.

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- Social preference between two alternatives *a* and *b* only depends on the voters' preferences between *a* and *b*.
- Given preference relations $P_1, \ldots, P_n \in \mathcal{P}$ and $Q_1, \ldots, Q_n \in \mathcal{P}$ then

$$\forall a\ b \in \mathcal{C}.\ (\forall i \in I.\ a \succ_{P_i} b \iff a \succ_{Q_i} b)$$
$$\Longrightarrow$$
$$(a \succ_{SWF(P_1,\ldots,P_n)} b \iff a \succ_{SWF(Q_1,\ldots,Q_n)} b)$$

Independence of Irrelevant Alternatives cont.

1	...	...	...	n	$\mathcal{F}$
a	a	c	...	a	...
⋮	c	⋮	...	⋮	a
c	b	...	a	c	...
⋮	⋮	⋮	c	⋮	c
b	...	a	b	...	b
⋮	...	b	...	b	...

Independence of Irrelevant Alternatives cont.

1	...	...	...	n	$\mathcal{F}$
c	c	c	...	a	...
⋮	a	⋮	...	⋮	c
a	b	...	a	c	...
⋮	⋮	⋮	c	⋮	a
b	...	a	b	...	b
⋮	...	b	...	b	...

Independence of Irrelevant Alternatives cont.

1	...	...	...	n	$\mathcal{F}$
c	c	c	a	a	...
⋮	a	⋮	c	⋮	c
a	b	...	...	c	...
⋮	⋮	⋮	⋮	⋮	a
b	...	a	b	...	b
⋮	...	b	...	b	...

Preliminary Lemma

Lemma (Strict Neutrality Lemma)

Consider two pairs $u, v \in \mathcal{C}$ *and* $a, b \in \mathcal{C}$. *Suppose all* i *have the same relation between* u, v *and* a, b ($u \succ v \iff a \succ b$). *Then the SWF ranks (strictly)* u, v *the same as* a, b.

Theorem (Arrow's Impossibility Theorem)

Any Social Welfare Function that respects transitivity, independence of irrelevant alternatives (IIA) and unanimity is dictatorial.

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Example (Naive Implementation)

1. Take two alternatives x and y and see which receives more votes.
2. Repeat for every pair.

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1. Take two alternatives x and y and see which receives more votes.
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Problem:

c	a	b
b	c	a
a	b	c

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Example (Naive Implementation)

1. Take two alternatives x and y and see which receives more votes.
2. Repeat for every pair.

Problem:

c	a	b	$c \succ b$
b	c	a	$b \succ a$
a	b	c	$a \succ c$
			$\lightning$

Examples cont.

Example (Allow Indifference)

c	a	b	$c \approx b$
b	c	a	$b \approx a$
a	b	c	

Examples cont.

Example (Allow Indifference)

c	a	a
b	c	b
a	b	c

Examples cont.

Example (Allow Indifference)

c	a	a	$a \succ b$
b	c	b	
a	b	c	

Examples cont.

Example (Allow Indifference)

c	a	a	$a \succ b$
b	c	b	$c \succ b$
a	b	c	

Examples cont.

Example (Allow Indifference)

c	a	a	$a \succ b$
b	c	b	$c \succ b$
a	b	c	$a \succ c$
			$\lightning$(IIA)

Proof

Theorem (Arrow's Impossibility Theorem)

Any Social Welfare Function that respects transitivity, independence of irrelevant alternatives (IIA) and unanimity is dictatorial.

Identify pivotal voter e

			$\mathcal{F}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	$\ldots$	$\ldots$	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	$\ldots$	$\ldots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Proof

Theorem (Arrow's Impossibility Theorem)

Any Social Welfare Function that respects transitivity, independence of irrelevant alternatives (IIA) and unanimity is dictatorial.

Identify pivotal voter e

1						n	$\mathcal{F}$
⋮	...	⋮	⋮	⋮	⋮	...	⋮
a	...	a	b	b	b	...	b
⋮	...	⋮	⋮	⋮	⋮	...	⋮
b	...	b	a	a	a	...	a
⋮	...	⋮	⋮	⋮	⋮	...	⋮

Proof

Theorem (Arrow's Impossibility Theorem)

Any Social Welfare Function that respects transitivity, independence of irrelevant alternatives (IIA) and unanimity is dictatorial.

Identify pivotal voter e

1						n	$\mathcal{F}$
⋮	…	⋮	⋮	⋮	⋮	…	⋮
a	…	a	a	b	b	…	b
⋮	…	⋮	⋮	⋮	⋮	…	⋮
b	…	b	b	a	a	…	a
⋮	…	⋮	⋮	⋮	⋮	…	⋮

Proof

Theorem (Arrow's Impossibility Theorem)

Any Social Welfare Function that respects transitivity, independence of irrelevant alternatives (IIA) and unanimity is dictatorial.

Identify pivotal voter e

1				e		n	$\mathcal{F}$
⋮	…	⋮	⋮	⋮	⋮	…	⋮
a	…	a	a	a	b	…	a
⋮	…	⋮	⋮	⋮	⋮	…	⋮
b	…	b	b	b	a	…	b
⋮	…	⋮	⋮	⋮	⋮	…	⋮

Proof cont.

Proof.

Show that pivotal voter e is a dictator ($\exists i.\ x \succ_i y \rightarrow x \succ y$)

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator ($\exists i.\ x \succ_i y \rightarrow x \succ y$)

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$

1			e			n	$\mathcal{F}$
⋮	…	⋮	⋮	⋮	…	⋮	
⋮	⋮	⋮	x	⋮	⋮	⋮	
⋮	…	⋮	⋮	⋮	…	⋮	
⋮	…	⋮	y	⋮	…	⋮	
⋮	…	⋮	⋮	⋮	…	⋮	

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator $(\exists i.\ x \succ_i y \rightarrow x \succ y)$

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$
- Take z such that $x \succ_e z \succ_e y$

1			e			n	$\mathcal{F}$
⋮	…	⋮	⋮	⋮	…	⋮	
⋮	⋮	⋮	x	⋮	⋮	⋮	
⋮	…	⋮	z	⋮	…	⋮	
⋮	…	⋮	y	⋮	…	⋮	
⋮	…	⋮	⋮	⋮	…	⋮	

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator ($\exists i.\ x \succ_i y \to x \succ y$)

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$
- Take z such that $x \succ_e z \succ_e y$

1			e			n	$\mathcal{F}$
z	…	z	⋮	⋮	…	⋮	
⋮	⋮	⋮	x	⋮	⋮	⋮	
⋮	…	⋮	z	⋮	…	⋮	
⋮	…	⋮	y	⋮	…	⋮	
⋮	…	⋮	⋮	z	…	z	

- $x \succ y$.

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator ($\exists i.\ x \succ_i y \rightarrow x \succ y$)

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$
- Take z such that $x \succ_e z \succ_e y$

1			e			n	$\mathcal{F}$
z	…	z	⋮	⋮	…	⋮	
⋮	⋮	⋮	x	⋮	⋮	⋮	$x \succ z$
⋮	…	⋮	z	⋮	…	⋮	
⋮	…	⋮	y	⋮	…	⋮	
⋮	…	⋮	⋮	z	…	z	

- $x \succ y$.

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator $(\exists i.\ x \succ_i y \rightarrow x \succ y)$

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$
- Take z such that $x \succ_e z \succ_e y$

1			e			n	$\mathcal{F}$
z	...	z	⋮	⋮	...	⋮	
x	x	x	x	x	x	x	$x \succ z$
⋮	...	⋮	z	⋮	...	⋮	
⋮	...	⋮	y	⋮	...	⋮	
⋮	...	⋮	⋮	z	...	z	

- $x \succ y$.

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator $(\exists i.\ x \succ_i y \rightarrow x \succ y)$

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$
- Take z such that $x \succ_e z \succ_e y$

1			e			n	$\mathcal{F}$
z	…	z	⋮	⋮	…	⋮	
⋮	⋮	⋮	x	⋮	⋮	⋮	$x \succ z$
⋮	…	⋮	z	⋮	…	⋮	$z \succ y$
⋮	…	⋮	y	⋮	…	⋮	
⋮	…	⋮	⋮	z	…	z	

- $x \succ y$.

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator $(\exists i.\ x \succ_i y \rightarrow x \succ y)$

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$
- Take z such that $x \succ_e z \succ_e y$

1			e			n	$\mathcal{F}$
z	...	z	⋮	⋮	...	⋮	
⋮	⋮	⋮	x	⋮	⋮	⋮	$x \succ z$
⋮	...	⋮	z	⋮	...	⋮	$z \succ y$
⋮	...	⋮	y	⋮	...	⋮	$x \succ y$
⋮	...	⋮	⋮	z	...	z	

- $x \succ y$.

□

Proof cont.

Proof.

Show that pivotal voter e is a dictator ($\exists i.\ x \succ_i y \rightarrow x \succ y$)

- Take arbitrary pair $x, y \in \mathcal{C}$ such that $x \succ_e y$
- Take z such that $x \succ_e z \succ_e y$

1			e			n	$\mathcal{F}$
z	...	z	⋮	⋮	...	⋮	
⋮	⋮	⋮	x	⋮	⋮	⋮	$x \succ z$
⋮	...	⋮	z	⋮	...	⋮	$z \succ y$
⋮	...	⋮	y	⋮	...	⋮	$x \succ y$
⋮	...	⋮	⋮	z	...	z	

- $x \succ y$. We identified an individual e who is a dictator

□

Formalization of Arrow's Theorem

Formalizations

- Mizar formalization by Freek Wiedijk
- Isabelle formalization by Nipkow
- Isabelle formalization by Peter Gammie

Further formalized results

Definition (Social Choice Function (SCF))

A social choice function is a mapping $\mathcal{G} : \mathcal{P} \times \cdots \times \mathcal{P} \to \mathcal{C}$

Theorem (Gibbard-Satterthwaite)

If $\mathcal{G}$ is incentive compatible and $|\mathcal{C}| \geq 3$ then $\mathcal{G}$ has a dictator.

Theorem (May's Theorem)

Majority rule is the only social choice function where $|\mathcal{C}| = 2$ and n is odd that is (1) anonymous (2) neutral (3) monotone.

Thanks for your attention

Lemma (Strict Neutrality Lemma)

Consider two pairs $u, v \in \mathcal{C}$ and $a, b \in \mathcal{C}$. Suppose all i have the same relation between u, v and a, b ($u \succ v \iff a \succ b$). Then the SWF ranks (strictly) u, v the same as a, b.

- assume u, v and a, b are not identical and $a \succeq b$
- Case $\{u, v, a, b\}$ are distinct

u	u	u	b	b	$a \succeq b$ by asm.
a	a	a	v	v	$u \succ a \wedge b \succ v$ by (U)
b	b	b	u	u	$u \succ v$
v	v	v	a	a	

- to show: $u \succ v$

□

Example (21 people rank 3 alternatives A, B and C)

1.	A	A	B	C	
2.	B	C	C	B	?
3.	C	B	A	A	
#votes	1	7	7	6	(21)