Complexity Analysis for Extended First Epsilon Theorem

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ε-calculus

- We consider a term construct by the ε operator, which looks like a quantifier as $\forall$ and $\exists$.
  - Example: $\varepsilon_x P(x, a)$.
  - Difference: ε forms a term, while $\forall$ and $\exists$ form a formula.
- $\varepsilon_x A(x)$ is intended to be a term, such that $A(\varepsilon_x A(x))$ holds.
- Classical first-order logic + ε is essentially same as the original first-order logic (Epsilon theorems).
- Existential and universal quantifiers are definable.

$$\exists_x A(x) := A(\varepsilon_x A(x)), \quad \forall_x A(x) := A(\varepsilon_x \neg A(x)).$$

ε-calculus as an extension of elementary calculus

- Naming conventions.
  - Free variables a, b, c
  - Bound variables x, y, z
  - Function symbols f, g
  - Predicate symbols P, Q
- Terms and formulas of EC (elementary calculus).

$$t, s ::= x \mid f(\vec{t})$$
$$A, B ::= P(\vec{t}) \mid \neg A \mid A \to B \mid A \wedge B \mid A \vee B$$

  Propositional reasoning available.

- Terms of ε-calculus EC_ε.

$$t, s ::= \ldots \mid \varepsilon_x A(x)$$

- *Critical axiom* $A(t) \to A(\varepsilon_x A(x))$, where a term t is arbitrary.

$\varepsilon_x A(x)$ occurs in critical axiom as above is *critical ε-term*.

Epsilon theorems

Hilbert and Bernays gave the following results.

Theorem (First and second epsilon theorems)

1. *Assume* $\mathbf{EC}_\varepsilon \vdash A$ *for a* $\forall, \exists, \varepsilon$*-free formula* A*. Then* $\mathbf{EC} \vdash A$*.*
2. *Assume* $\mathbf{PC}_\varepsilon \vdash A$ *for a* ε*-free formula* A*. Then* $\mathbf{PC} \vdash A$*.*

Theorem (Extended first epsilon theorem)

Assume $\mathbf{EC}_\varepsilon \vdash A(\vec{e})$ *for a* $\forall, \exists$*-free formula* $A(\vec{a})$ *and* ε*-terms* $\vec{e}$*.*
Then $\mathbf{EC} \vdash \bigvee_{i=1}^{n} A(\vec{s}_i)$ *for some* ε*-free terms* $\vec{s}_1, \ldots, \vec{s}_n$*.*

- Herbrand's theorem as a corollary of extended first epsilon theorem.
- n gives the complexity of ε-elimination procedure.
- Equality cases ($\mathbf{EC}_\varepsilon^=$, $\mathbf{PC}_\varepsilon^=$) were covered due to Hilbert and Bernays.

Motivation

We study proof complexity analysis for ε-calculus with equality.

- Finding the complexity of the critical axiom via analyzing the method of eliminating it, i.e. the ε-substitution method.

Content of this talk.

- (Rest of) introduction to ε-calculus with equality ($=$).
- Complexity analysis for extended first epsilon theorem (j.w.w. Georg Moser).

On the formula of the critical axiom

- Recall the critical axiom $A(t) \to A(\varepsilon_x A(x))$.
- We want to determine whether a given formula B is an instance of $A(\varepsilon_x A(x))$.
  1. Find a list of ε-terms $\varepsilon_x B_1(x), \ldots, \varepsilon_x B_k(x)$ occurring in B.
  2. For each $\varepsilon_x B_i(x)$, check whether $B \equiv B_i(\varepsilon_x B_i(x))$.
  3. If such an i is found, the answer is yes and $B_i(x)$ is the solution for $A(x)$.
  4. Otherwise, the answer is no.
- This algorithm was implemented in the proof assistant Minlog.
- At the step 3, we can discard the rest $\varepsilon_x B_{i+1}(x), \ldots, \varepsilon_x B_k(x)$.

Lemma

$\varepsilon_x B_i(x) \not\equiv \varepsilon_x B_j(x) \to B_i(\varepsilon_x B_i(x)) \not\equiv B_j(\varepsilon_x B_j(x))$.

Critical axiom in intuitionistic EC

Intuitionistic (even minimal) EC extended by the critical axiom proves formulas not provable in intuitionistic predicate logic. Aguilera and Baaz (2016) found that the following axioms characterize intuitionistic calculus extended by epsilon.

- $(A \to \exists_x B(x)) \to \exists_x (A \to B(x))$
- $(\forall_x A(x) \to B) \to \exists_x (A(x) \to B)$
- $\forall_x (A \lor B(x)) \to A \lor \forall_x B(x)$

Examples of EC$_\varepsilon$-proofs

1. $(A \to \exists_x B(x)) \to \exists_x (A \to B(x))$, where $x \notin FV(A)$.
   Proof. By definition, we prove
   $(A \to B(\varepsilon_x B(x))) \to A \to B(\varepsilon_x (A \to B(x)))$.
   Let $C(x)$ be $A \to B(x)$, then our goal is by the critical axiom
   $C(\varepsilon_x B(x)) \to C(\varepsilon_x C(x))$.
2. $\exists_x (A(x) \to \forall_y A(y))$
   Proof. By definition, we prove
   $A(\varepsilon_x (A(x) \to A(\varepsilon_x \neg A(x)))) \to A(\varepsilon_x \neg A(x))$.
   Let $C(x)$ be $A(x) \to A(\varepsilon_x \neg A(x))$, the critical axiom for $C(x)$
   proves $\{A(\varepsilon_x \neg A(x)) \to A(\varepsilon_x \neg A(x))\} \to A(e) \to A(\varepsilon_x \neg A(x))$,
   where $e = \varepsilon_x (A(x) \to A(\varepsilon_x \neg A(x)))$.

ε-substitution for first epsilon theorem

Assume a proof of a formula E containing a critical axiom.

$$A(t) \to A(\varepsilon_x A(x))$$

1. Proof $\bar{\pi}$ of $\neg A(t) \to E$.
   - Easy due to ex falso quodlibet.
2. Proof π of $A(t) \to E$.
   - Replace $\varepsilon_x A(x)$ in the original proof by t.
   - $A(t') \to A(t)$ is provable due to the assumption $A(t)$.
3. E holds due to $\neg A(t) \vee A(t)$.

We briefly discuss the termination issue later.

ε-substitution for extended first epsilon theorem

Assume a proof of a formula $E(\varepsilon_x A(x))$ containing a critical axiom.

$$A(t) \to A(\varepsilon_x A(x))$$

1. Proof $\bar{\pi}$ of $\neg A(t) \to E(\varepsilon_x A(x))$.
   - Easy due to ex falso quodlibet.
2. Proof π of $A(t) \to E(t)$.
   - Replace $\varepsilon_x A(x)$ in the original proof by t.
   - The original goal formula goes to $E(t)$.
   - $A(t') \to A(t)$ is provable due to the assumption $A(t)$.
3. $E(\varepsilon_x A(x)) \lor E(t)$ holds due to $\neg A(t) \lor A(t)$.

ε-substitution for $\mathbf{EC}_\varepsilon$

Assume a proof containing critical axioms.

$$
\begin{gathered}
A_1(t_{11}) \to A_1(\varepsilon_x A_1(x)) \\
\vdots \\
A_1(t_{1n_1}) \to A_1(\varepsilon_x A_1(x)) \\
A_2(t_{21}) \to A_2(\varepsilon_x A_2(x)) \\
\vdots \\
A_m(t_{mn_m}) \to A_m(\varepsilon_x A_m(x))
\end{gathered}
$$

- In order to make the process of replacement terminating, we introduce the *rank* and *degree*.
- We replace from the ε-term with the greatest rank r and its degree is greatest among ε-terms of rank r in the proof.

Rank of formulas, terms, proofs

Rank (rk) defined for terms, formulas and proofs as follows.

$\mathrm{rk}(a) := \mathrm{rk}(x) := 0, \quad \mathrm{rk}(f\vec{t}\,) := \max\{\mathrm{rk}(t_i) \mid i < |\vec{t}\,|\},$
$\mathrm{rk}(\varepsilon_x A) := \max\{\mathrm{rk}(t) \mid t \text{ a subterm of } \varepsilon_x A \text{ s.t. } x \text{ occurs in } FV(t)\} + 1,$
$\mathrm{rk}(A) := \max\{\mathrm{rk}(t) \mid t \text{ is an immediate subterm of } A\},$
$\mathrm{rk}(\pi) := \max\{\mathrm{rk}(e) \mid e \text{ is a critical } \varepsilon\text{-term in } \pi\}.$

Example

$\mathrm{rk}(\varepsilon_x P(t, \varepsilon_y P(y, x))) = 2$ *where* x *does not occur in* $FV(t)$.

Lemma

$\mathrm{rk}(\varepsilon_x A(x, a)) = \mathrm{rk}(\varepsilon_x A(x, t))$ *for any term* t.

Degree of formulas, terms, proofs

$$\deg(a) := \deg(x) := 0, \quad \deg(f\vec{t}\,) := \max\{\deg(t_i) \mid i < |\vec{t}\,|\},$$
$$\deg(\varepsilon_x A) := \deg(A) + 1,$$
$$\deg(A) := \max\{\deg(t) \mid t \text{ is a bound variable-free subterm of A}\},$$
$$\deg(\pi) := \max\{\deg(e) \mid e \text{ is a critical } \varepsilon\text{-term in } \pi\}.$$

Example

$$\deg(\varepsilon_x P(a, \varepsilon_y P(y, x))) = \deg(a) = 1.$$

Complexity analysis for $EC_\varepsilon^=$

1. Results for EC_ε by Moser and Zach in 2006
2. The rest of this talk is about a progress in $EC_\varepsilon^=$

For $EC_\varepsilon^=$, formulas are extended as

$$A, B ::= \ldots \mid s = t$$

Additional axioms.

- $t = t$
- $s = t \to P(s, \vec{u}) \to P(t, \vec{u})$ (for any predicate symbol P)
- $s = t \to f(s, \vec{u}) = f(t, \vec{u})$
- $s = t \to \varepsilon_x A(x, s, \vec{u}) = \varepsilon_x A(x, t, \vec{u})$
  for an *ε-matrix* $\varepsilon_x A(x, a, \vec{b})$ (see the next slide)

ε-matrices

An ε-term $\varepsilon_x A(x, \vec{b})$ is an ε-*matrix* if

- Its immediate subterms are free variables $\vec{b}$.
- Each b_i occurs exactly once.

Lemma

$s = t \to A(s) \to A(t)$ in $\mathbf{EC}^{=}_{\varepsilon}$.

- The restriction due to ε-matrices simplifies the matter.
  - $s = t \to \varepsilon_x A(x, \varepsilon_y B(y, s)) = \varepsilon_x A(x, \varepsilon_y B(y, t))$ is not allowed to be an ε-equality axiom formula.
  - If it was, a substitution $\{\varepsilon_y B(y, s) \mapsto r\}$ makes a problem: $s = t \to \varepsilon_x A(x, r) = \varepsilon_x A(x, \varepsilon_y B(y, t)) \ \cdots$??

ε-substitution for $\mathbf{EC}_{\varepsilon}^{=}$

Assume a proof containing critical axioms and ε-equality axioms.

$$A(t_1, v) \to A(\varepsilon_x A(x, v), v)$$
$$\vdots$$
$$A(t_n, v) \to A(\varepsilon_x A(x, v), v)$$
$$u_1 = v \to \varepsilon_x A(x, u_1) = \varepsilon_x A(x, v)$$
$$\vdots$$
$$u_m = v \to \varepsilon_x A(x, u_m) = \varepsilon_x A(x, v)$$

1. Proof $\bar{\pi}$ of $(\bigwedge \neg u_i = v) \to E$.
   - ε-equality formulas go provable due to ex falso quodlibet.
   - Critical axioms managed by ε-substitution for $\mathbf{EC}_{\varepsilon}$.
2. Proof π_i of $u_i = v \to E$.
   - Replace $\varepsilon_x A(x, v)$ by $\varepsilon_x A(x, u_i)$. i-th equality formula is trivial.
   - For $j \neq i$, $u_j = v \to A(x, u_j) = \varepsilon_x A(x, u_i)$ is provable.
   - For all k we still need to manage $A(t'_k, v) \to A(\varepsilon_x A(x, u_i), v)$.

Key lemma

Prove $u = v \to A(t, v) \to A(\varepsilon_x A(x, u), v)$. Assume $u = v$.

1. $A(t, v) \to A(t, u)$.
2. $A(t, u) \to A(\varepsilon_x A(x, u), u)$ (critical axiom).
3. $A(\varepsilon_x A(x, u), u) \to A(\varepsilon_x A(x, u), v)$.

Lemma

Assume $\varepsilon_x A(x, b)$ is an ε-matrix. There is an $\mathbf{EC}_\varepsilon^=$-proof π of $u = v \to A(t, u) \to A(t, v)$ such that $\mathrm{cc}(\pi) \leq \deg(A(a, b))$ and $\mathrm{rk}(\pi) \leq \mathrm{rk}(A(a, b))$.

Here, $\mathrm{cc}(\pi)$ is the number of critical and ε-equality axioms in π.

Complexity analysis for one step ε-substitution

Assume from an $\mathbf{EC}_{\varepsilon}^{=}$-proof π of $E(e)$ we obtained π_e of $\bigvee_i E(s_i)$.

Lemma

The complexity of π_e *is as follows. Let* r *be* $\mathrm{rk}(\pi)$.

- $\mathrm{wd}(\pi_e, r) \leq 4 \cdot \mathrm{wd}(\pi, r)^2 + 2 \cdot \mathrm{wd}(\pi, r)$.
- $\mathrm{cc}(\pi_e) \leq (\mathrm{cc}(\pi) + 1) \cdot (\mathrm{wd}_\pi(e) + 1)$.
- $\mathrm{wd}(\pi_e, r') \leq (\mathrm{wd}(\pi, r') + \mathrm{wd}(\pi, r)) \cdot (\mathrm{wd}(\pi, r) + 1)$ *for* $r' < r$.

Here,

- $\mathrm{wd}_\pi(e)$ *is the number of distinct critical and* ε*-equality axioms belonging to* e.
- $\mathrm{wd}(\pi, r) := \max\{\mathrm{wd}_\pi(e) \mid \mathrm{rk}(e) = r \,\&\, e \text{ critical } \varepsilon\text{-term in } \pi\}$

Future work

- It is worth to further investigate a way to keep the proof of the key lemma simple.
    - Consider an intermediate system $\mathbf{EC}_{\varepsilon}^{=} \cup \mathbf{EC}^{=}$.
- Give the bound of the length of disjunction for extended first epsilon theorem in $\mathbf{EC}_{\varepsilon}^{=}$.