Legendrian Knots and First Order Theorem Provers

T. V. H. Prathamesh (Joint work with Dheeraj Kulkarni)

15-11-2017

Background

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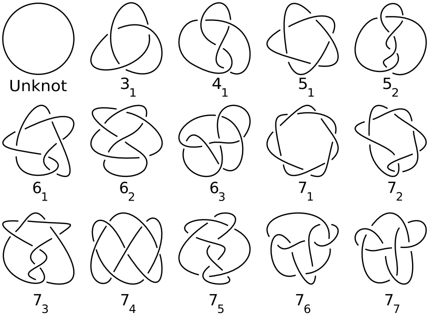
- In this talk, we look at how they proved effective in constructing a new topological invariant called Legendrian racks.
- Legendrian racks are an invariant of Legendrian knots, an object of study in contact topology.

Knot



(Image source: Wikipedia)

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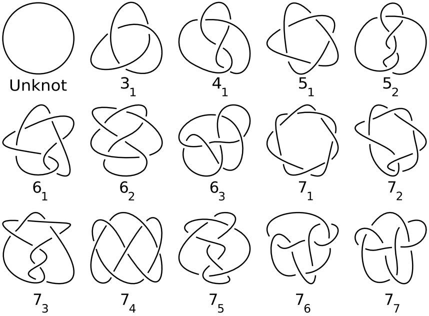

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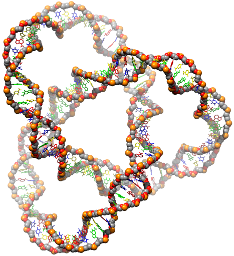
Definition (Formal Definition)

A knot K is the image of a smooth, injective map $h : S^1 \to \mathbb{R}^3$ such that, $h'(\theta) \neq 0$ for all $\theta \in S^1$.

Why study knots

DNA Structures



Quantum Knots

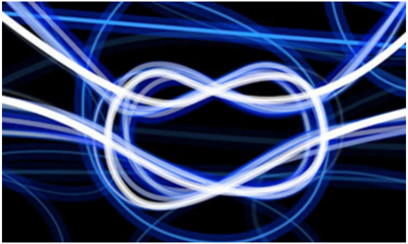


Image source(s): thinglink.com, edx.org

Question: **What do we study about knots?**

When are two knots (or links) regarded as same or different?

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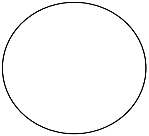

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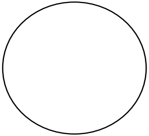

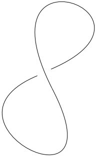

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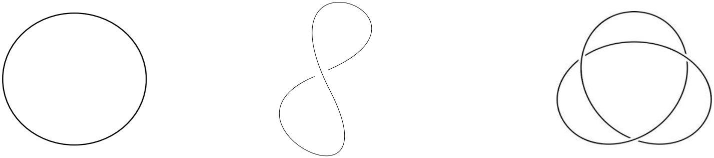

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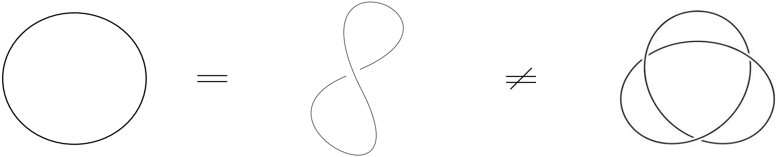
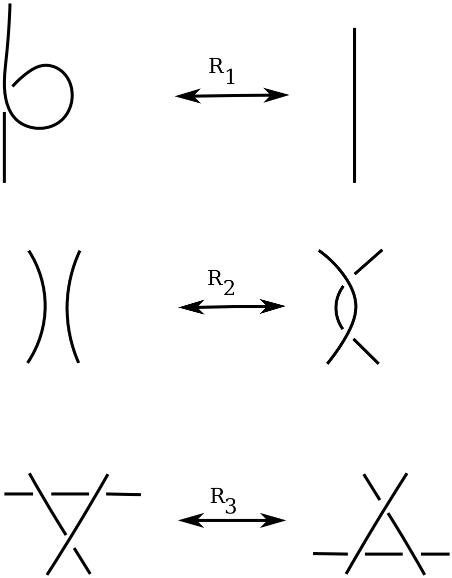


Knot Equivalence

- Two knots are said to be the same if one can be obtained from another without involving any cutting and pasting.
- The formal definition is in terms of 'ambient isotopy'.
- In practise, we use a 'combinatorial' version of equivalence, called Reidemeister moves.

Reidemeister moves



How do show in/equivalence of two knots?

- Run a knot recognition algorithm.
  - Drawbacks: Complexity is exponential in number of crossings and very few implementations available.
- Compute an invariant. If the values differ, they are different.
  - Drawback: Already tried for most cases. Complexity bounds of the difficult invariants are not reliably known.
- Use an automated theorem prover [Fish, Lisista (2014)].
  - Drawback: Early stages of research, no new results have been derived using this method.

How does one achieve the above for knots?

Topological Objects

Quandles

Combinatorial/Algebraic Structures

How does one achieve the above for knots?

Topological Objects

↓ Quandles

Combinatorial/Algebraic Structures

↓ First Order Logic

Automated Theorem Provers

Some definitions

- Left Self Distributivity:
  $\forall x.\forall y.\forall z.\ (x * y) * z = (x * z) * (y * z)$.
- Left Inverse: $\forall y.\forall z.\exists! x.\ z = (x * y)$.
- Idempotence: $\forall x.\ x * x = x$.

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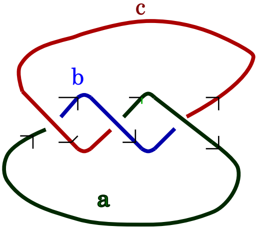
Definition

- A **rack** $(R, *)$ is a model of the left self distributive and left inverse axioms.
- A **quandle** is an idempotent rack.
- **Remark:** We can use z/y to denote x in the left inverse axiom.

Knots and Quandles

Knot



- $a * c = b$.
- $b * a = c$.
- $c * b = a$.

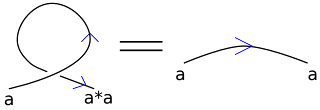
Quandle

- $(S, *)$.
- Generators: Strands.
- Generating Relations: Crossings.
- $(S, *)$:
  1. Closed under $(*)$.
  2. Left self-distributive.
  3. Left inverse property.
  4. Idempotent

Knots and Quandles

Knot

- Left self distributivity to third Reidemeister move.
- Left inverse corresponds to second Reidemeister move.
- Idempotence corresponds to third Reidemeister move.



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Automated Theorem Proving and Quandles

- The quandle corresponding to the unknot is the trivial quandle.
- Checking if a given knot is an unknot, equals checking triviality of the quandle.
- This can be achieved by an ATP for FOL.

Dheeraj asked:

Question: Can Automated theorem provers help distinguish Legendrian knots?

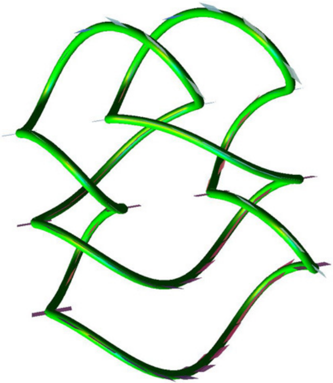
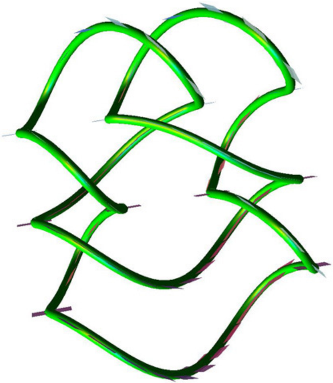


Image Source: Lenny Ng

Legendrian Knots



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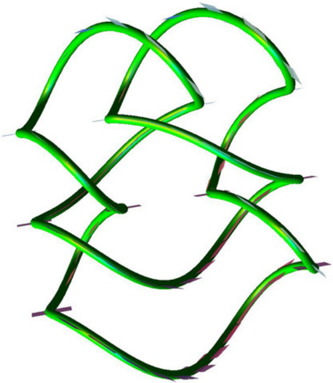
- Legendrian knots occur in contact geometry.

Legendrian Knots



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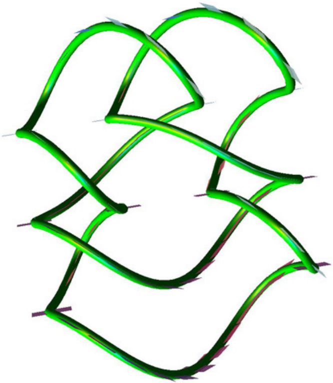

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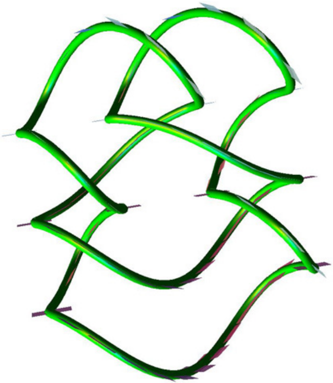

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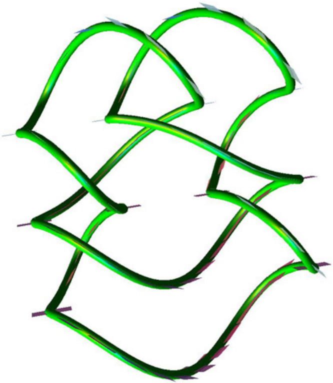

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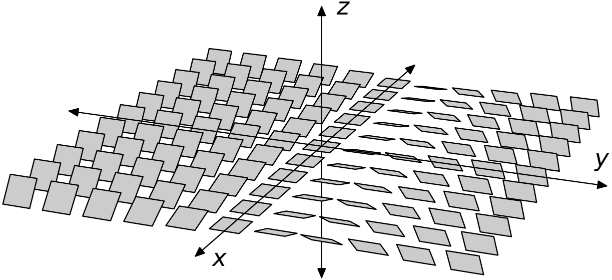
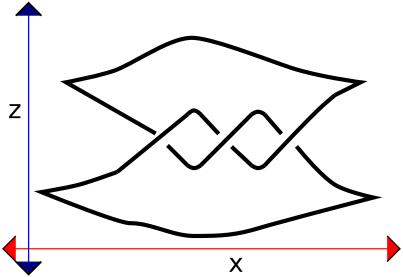
- They help distinguishing geometric structures called contact manfolds.
- Contact manifolds arise out of optics and related PDE's.
- They are odd dimensional analogues of 'symplectic manifolds'.
- Legendrian knots can be understood as knots with conditions on tangency.

Standard Contact Form on $\mathbb{R}^3$

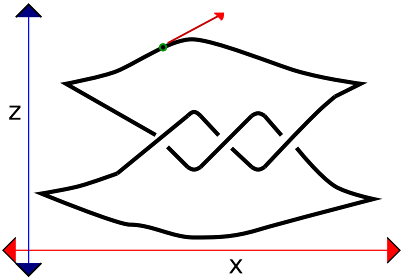
- Point : $(x,\ y,\ z)$
- Embedded Plane: $\{(u,\ v,\ y \cdot u) \mid u, v \in \mathbb{R}\}$



Only admissible paths in such a space, are those whose tangents at each lie in the associated embedded plane.



- $y = dz/dx$.
- No vertical tangencies in the front projection.
- Vertical tangencies are replaced by cusps.
- Only one kind of crossing is present.
- Knot Eq $\prec_{weaker}$ Legendrian knot Eq.



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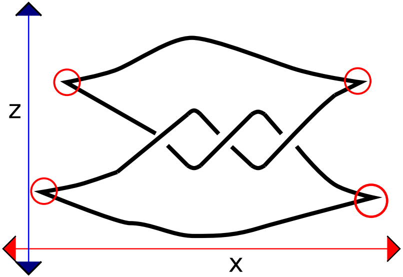

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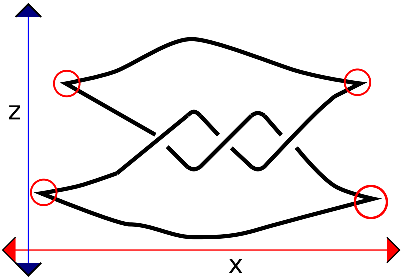

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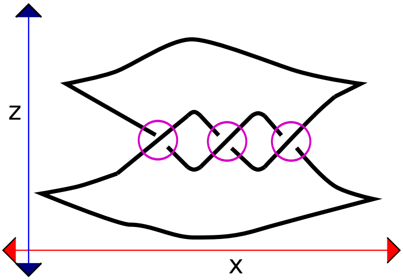

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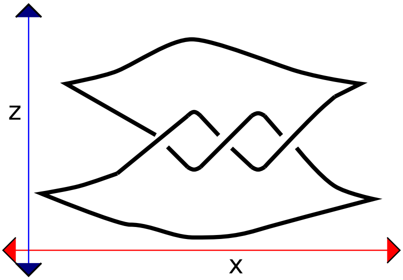

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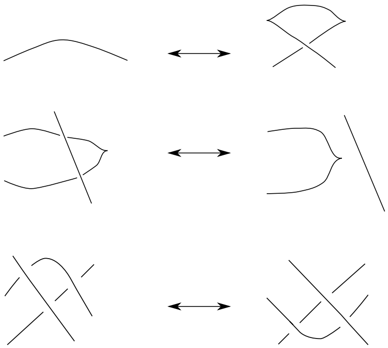
When are two Legendrian knots *same*?

- Equivalence between Legendrian knots is formally defined in terms of *Legendrian isotopy*. The definition involves derivatives and ambient isotopy.

- It can be pictorially understood using Legendrian Reidemeister moves.

Legendrian Reidemeister moves



Legendrian knots and knots

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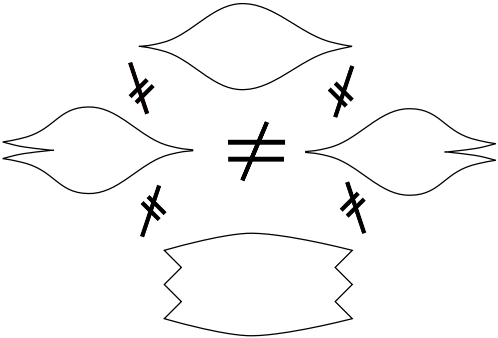


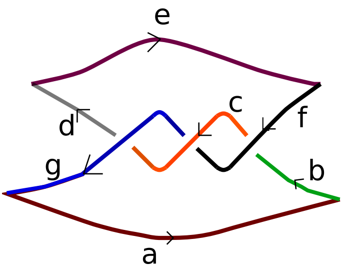
Distinguishing Legendrian Knots

- Few knot invariants: Chekanov DGA, Rotation number, Tb number Et Al.
- Some of them can be computed using regular computer programs. Though not sufficiently high number of Legendrian knots are classified.
- However, none of these invariants are implementable in an Automated Theorem Prover.

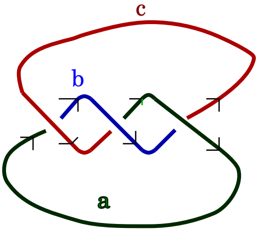
Motivating Task: To find quandle like invariants of Legendrian knots.

We discovered such a class of invariants, which we call Legendrian racks.

Legendrian Knot

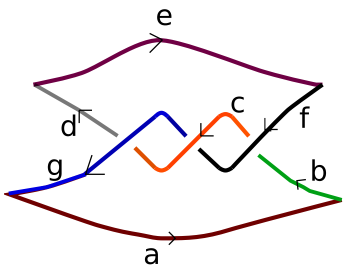


(Classical) Knot



Legendrian Knots and Legendrian Racks

Legendrian Knot



- $b * f = c$.
- $b = a^{n+1}$.

Legendrian racks

- $\{(LR_n, *)\}$
- Generators: Strands.
- Generating Relations: Crossings + Cusps.
- $(LR_n, *)$:
  1. Closed under $(*)$.
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  4. $\forall x.\ x^{(2n+2)} = x$.

How did we discover it

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Legendrian Reidemeister moves were formalized as first-order logic axioms in the language - $(*, /, U, D)$.

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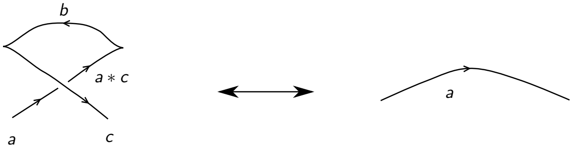
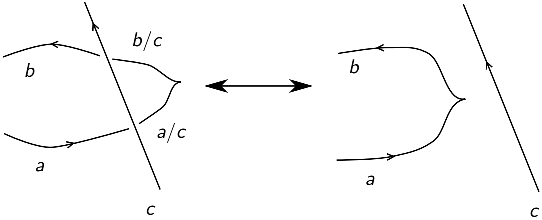


$$U(a * c, b) \wedge D(b, c) \longleftrightarrow (a = c) \wedge (b = c)$$



$$U(a, b) \longleftrightarrow U(a/c, b/c)$$

A note on the axioms

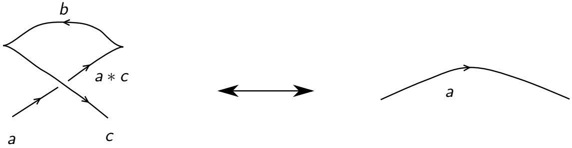
- To each Legendrian knot, we associate the canonical model of these axioms.
- These axioms consisted of rack axioms + axioms for Legendrian Reidemeister moves. The idempotence axioms does not hold true.
- The third Legendrian Reidemeister move does not involve predicates and follows from the rack axioms.
- There were a net total of 18 axioms to account for various geometric possibilities.

We ran a few experiments on Prover9, to check the following facts:

- $U(a, b) = D(a, b)$
- $U(a, b) \longrightarrow (a = b)$.

Both of these returned TRUE, which implies that the structure did not detect 'cusps'.

Revisiting axioms



$$U(a * c, b) \wedge D(b, c) \iff (a = c) \wedge (b = a^{\square})$$

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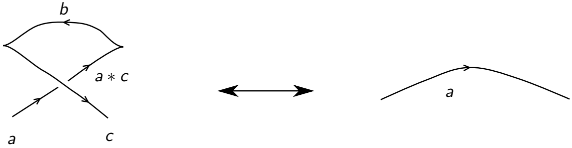

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New axiom (schema):

$$U_n(a * c, b) \wedge D_n(b, c) \iff (a = c) \wedge (b = a^n)$$

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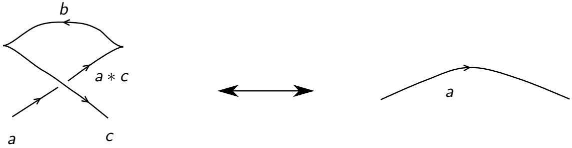

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The experiments were repeated for values of $n = 1, ..., 10$.

Results

- $U_n(a, b) = D_n(a, b)$.
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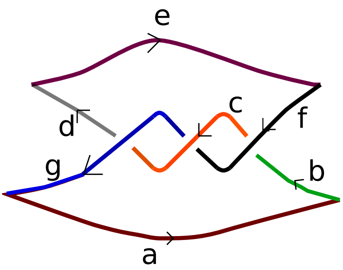
Results and observations:

- The first case always returned TRUE. Thus orientation is not detected.
- The second case was FALSE, except when n is of the form 2^k.

Experiment: $U_n(a, b) \wedge D_n(b, a) \implies b = a$.

- Above holds true for all tested n.
- In the proof, we noticed the following result $U_n(a, b) \longleftrightarrow (b = a^n)$.
- By carefully going through proofs for values of n, we were able to extract general proofs of the above results $\forall n \in \mathbb{N}$.
- We subsequently reworked the axioms by substituting for $U_n(a, b)$.

Legendrian Knot



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Legendrian racks

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- Generators: Strands'.
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Quandles and Legendrian Racks

`Quandles`

- Generators: Strands.
- Generating Relations: Crossings.
- $(S, *)$:
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  4. Idempotent
- $K \to (Q(K), *)$.
- Comlete invariant.

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  4. $\forall x.\ x^{(2n+2)} = x$.
- $LK \to \{(LR_n(LK), *)\}_{n \in \mathbb{N}}$.
- Not a comlete invariant.

Quandles and Legendrian Racks

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  4. $\forall x.\ x^{(2n+2)} = x$.
- $LK \rightarrow \{(LR_n(LK), *)\}_{n\in\mathbb{N}}$.
- Invariant, but not a complete invariant.

Proof of invariance

Theorem

For each $n \in \mathbb{N}$, LR_n is an invariant of Legendrian rack moves.

Proof.

Check for each Legendrian Reidemeister move. We get a set of identities, which are proved by a combination of induction and symbolic computations. □

Remark

Most of the proofs obtained again involved using automated theorem provers for specific cases, and then subsequently generalized the automatically generated proofs.

Detecting different Legendrian unknots

Input: Generating relations of various Legendrian unknots.
Conjecture: Triviality of the rack LR_n.

Output: Finite model counterexamples were generated, when number of strands did not equal 2^k, and for values of LR_n, where $(n, s_L) > 1$ (s_L is the minimum number of strands).

The finite models generated on closer look, revealed a pattern:

- There was a link between values of n, for which counterexamples were generated, and number of strands.
- Cardinality of the models was always a prime number..
- They were equal to a known example of racks, called the coloring racks $\{(C_n, *)\}$.
- Cardinality related to a known invariant called the Thurston-Bennequin number, tb.

Classifying Legendrian unknots

Theorem

If two Legendrian knots have the same tb number, each n-Legendrian rack is isomorphic.

Theorem

Given two Legendrian unknots L_1 and L_2 , such that there exists an odd prime p and $k \in \mathbb{N}$, such that:

- $p^k | tb(L_1)$,
- $p^k \not| tb(L_2)$,

then $\exists n.\ LR_n(L_1) = LR_n(L_2)$.

Proof.

There exists a surjective homomorphism in the first instance to C_{p^k}, which does not hold true in the second case. □

Further Questions

- As of yet, little is known about these invariants in more general cases..
- Is the word problem decidable in a Legendrian rack?
- Is the isomorphism problem decidable?
- Term rewriting related questions: Existence of unique normal forms, completion...
- Relationship with other known invariants, for more general cases.

Thank You