Computational Logic

# The Perron–Frobenius Theorem for Certification of Complexity Proofs

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# Overview

- Introduction
- Certifying Matrix Growth
- Proofs
- Formalization

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# Certification Approach

- Take input problem, e.g., Boolean formula
- Analyse using automated untrusted tools, e.g., run SAT-solver
- Obtain answer (SAT/UNSAT) + certificate
- Check certificate by trusted tool (trusted = formal proof)
  - Certification can be easy
    positive answer of SAT-solver with assignment $x, \neg y, \neg z, \ldots$
  - Certification can be hard or expensive
    "there is no satisfying assignment", DRAT, ...

# Complexity of Term Rewrite Systems

$$\begin{aligned}
\mathsf{sort}(\mathsf{Cons}(x, xs)) &\to \mathsf{insort}(x, \mathsf{sort}(xs)) \\
\mathsf{sort}(\mathsf{Nil}) &\to \mathsf{Nil} \\
\mathsf{insort}(x, \mathsf{Cons}(y, ys)) &\to \mathsf{Cons}(x, \mathsf{Cons}(y, ys)) && \mid x \leqslant y \\
\mathsf{insort}(x, \mathsf{Cons}(y, ys)) &\to \mathsf{Cons}(y, \mathsf{insort}(x, ys)) && \mid x \not\leqslant y \\
\mathsf{insort}(x, \mathsf{Nil}) &\to \mathsf{Cons}(x, \mathsf{Nil})
\end{aligned}$$

Aim: bound on maximal number of rewrite steps starting from

$$\mathsf{sort}(\mathsf{Cons}(x_1, \ldots \mathsf{Cons}(x_n, \mathsf{Nil})))$$

# Running automated complexity tool

Running TCT on TRS yields $\mathcal{O}(n^2)$ + certificate

$$[\![\mathsf{sort}]\!](xs) = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot [\![xs]\!]$$

$$[\![\mathsf{insort}]\!](x, xs) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot [\![xs]\!] + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$[\![\mathsf{Cons}]\!](x, xs) = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \cdot [\![xs]\!] + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$[\![\mathsf{Nil}]\!] = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

# Certification — part 1

Obtain strict decrease in every rewrite step:

$$[\![\mathsf{sort}(\mathsf{Cons}(x, xs))]\!] = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot [\![xs]\!] + \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \begin{matrix} > \\ \geq \\ \geq \end{matrix} \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot [\![xs]\!] + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = [\![\mathsf{insort}(x, \mathsf{sort}(xs))]\!]$$

# Certification — part 2

Bound initial interpretation:

$$[\![\mathsf{sort}(\mathsf{Cons}(x_1, \dots \mathsf{Cons}(x_n, \mathsf{Nil})))]\!] = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \left( A^n \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \sum_{i<n} A^i \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right) \in \mathcal{O}(n \cdot A^n)$$

Key analysis: growth of values of $A^n$ depending on $n$

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# Eigenvalues and eigenvectors

Matrix $A$ has eigenvector $v \neq 0$ with eigenvalue $\lambda$ if

$$Av = \lambda v$$

Remark

- $\lambda$ is eigenvalue of $A$ if and only if
  $\lambda$ is root of characteristic polynomial $\chi_A$

Consequences

- $A^n v = \lambda^n v$
- $|A^n v| = |\lambda|^n |v|$
- if $|\lambda| > 1$ then $A^n$ grows exponentially

**Theorem**

*$A^n$ grows polynomially if and only if*
*$|\lambda| \leqslant 1$ for all eigenvalues $\lambda$ of $A$*

# Jordan blocks

Matrix $A$ has Jordan normal form $J = PAP^{-1}$

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_k \end{pmatrix} \qquad J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda_i & 1 \\ 0 & \dots & \dots & 0 & \lambda_i \end{pmatrix}$$

Remarks

- the $\lambda_i$s are precisely the eigenvalues of $A$
- $\chi_A = \chi_J = \prod_{i=1}^{k}(x - \lambda_i)^{s_i}$ where $s_i$ = size of Jordan block $J_i$
- $J$ is unique up to permutation of blocks

Consequences

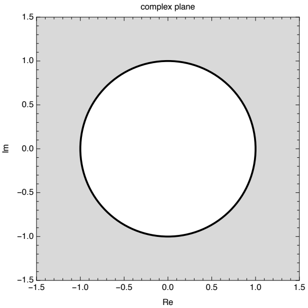
- $A^n = P^{-1}J^nP$, so $A$ and $J$ have same growth rate
- $J_i^n \in \Theta(n^{s_i-1}|\lambda_i|^n)$
- if $\max_i |\lambda_i| = 1$ then $A^n \in \Theta(n^{s-1})$ where $s = \max_{|\lambda_i|=1} s_i$

# Basic certification algorithm for $A^n \in \mathcal{O}(n^d)$

Input: Matrix $A$ and degree $d$

Output: Accept or assertion failure.

1. Compute all eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ (all complex roots of $\chi_A$)
2. Compute spectral radius $\rho_A := \max_i |\lambda_i|$
3. Assert $\rho_A \leqslant 1$
4. For each $\lambda_i$ with $|\lambda_i| = 1$, and Jordan block of $A$ and $\lambda_i$ with size $s_i$, assert $s_i - 1 \leqslant d$
5. Accept



# Example insertion sort

Input: Matrix $A$ and degree $d$

Output: Accept or assertion failure.

1. Compute all eigenvalues $\lambda_1, \dots, \lambda_n$ of $A$ (all complex roots of $\chi_A$)
2. Compute spectral radius $\rho_A := \max_i |\lambda_i|$
3. Assert $\rho_A \leqslant 1$
4. For each $\lambda_i$ with $|\lambda_i| = 1$, and Jordan block of $A$ and $\lambda_i$ with size $s_i$, assert $s_i - 1 \leqslant d$
5. Accept

$$\text{Input: } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, d = 1$$

1. $\lambda_1 = 1, \lambda_2 = 0$
2. $\rho_A = 1$

4. $s_1 - 1 = 2 - 1 \leqslant 1 = d$

# Another example

Input: $A = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

1. $\chi_A = \dfrac{(x-1)\,(8x^3 - 4x^2 - 2x - 1)}{8}$

$$\lambda_1 = 1$$

$$\lambda_2 = (\text{root } \#1 \text{ of } f_1)$$

$$\lambda_3 = (\text{root } \#1 \text{ of } f_2) + (\text{root } \#1 \text{ of } f_3)\mathrm{i}$$

$$\lambda_4 = (\text{root } \#1 \text{ of } f_2) + (\text{root } \#2 \text{ of } f_3)\mathrm{i}$$

$$f_1 = 8x^3 - 4x^2 - 2x - 1$$

$$f_2 = 32x^3 - 16x^2 + 1$$

$$f_3 = 1024x^6 + 512x^4 + 64x^2 - 11$$

# The problem and its solution

- algorithm 1 requires precise calculations ($|\lambda_i| = 1$)
- precise calculations with algebraic numbers are expensive
- aim: avoid explicit computation of eigenvalues
- solution: apply the Perron–Frobenius theorem

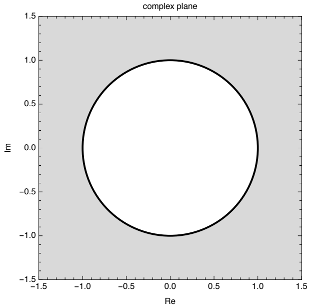
# Perron–Frobenius, Part 1

## Theorem (Perron–Frobenius)

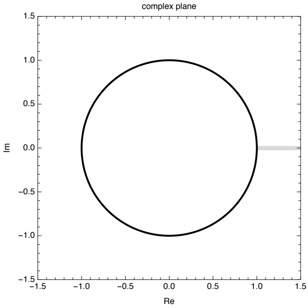
*Let $A$ be a non-negative real matrix*

- $\rho_A$ *is an eigenvalue of* $A$

## Consequence




$\longrightarrow$



# Perron–Frobenius, Part 2

**Theorem (Perron–Frobenius)**

*Let $A$ be a non-negative real and irreducible matrix*

- $\rho_A$ *is an eigenvalue of* $A$
- $\rho_A$ *has multiplicity 1*
- $\exists f\ k.\ \chi_A = f \cdot (x^k - \rho_A^k) \wedge (f(y) = 0 \longrightarrow |y| < \rho_A)$
- $\ldots$

Consequence

- non-negative real and irreducible matrices have constant or exponential growth

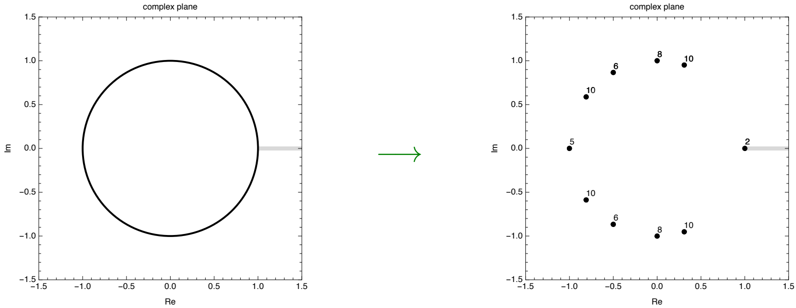
# Perron–Frobenius, Part 3

## Theorem

*Let $A$ be a non-negative real matrix*

- $\rho_A$ *is an eigenvalue of* $A$
- $\exists f\ K.\ \chi_A = f \cdot \prod_{k \in K}(x^k - \rho_A^k) \wedge (f(y) = 0 \longrightarrow |y| < \rho_A)$

## Consequence



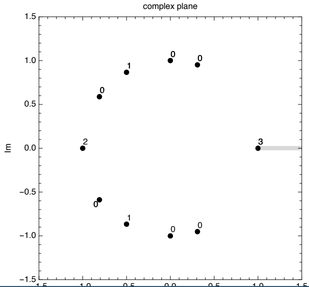
# Uniqueness of f and K

**Theorem**

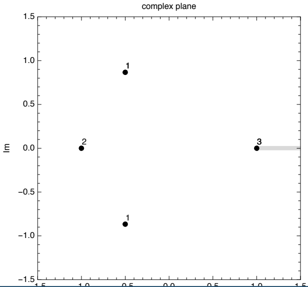
*Let $A$ be a non-negative real matrix*

- $\rho_A$ *is an eigenvalue of* $A$
- $\exists! f\, K.\ \chi_A = f \cdot \prod_{k \in K}(x^k - \rho_A^k) \wedge (f(y) = 0 \longrightarrow |y| < \rho_A)$
- decompose $\chi_A$ *computes* $f$ *and* $K$ *for* $\rho_A = 1$

## Consequence




$\xrightarrow{K=\{2,2,3\}}$



# New certification algorithm

Input: non-negative matrix $A$ and degree $d$

Output: Accept or assertion failure.

1. Assert that $\chi_A$ has no real roots in $(1, \infty)$ via Sturm's method
2. Compute $K$ via decompose $\chi_A$
3. If $|K| - 1 \leqslant d$ then accept
4. Check the Jordan blocks for eigenvalue 1, i.e., assert that each Jordan block of $A$ and 1 has size $s \leqslant d + 1$
5. If $\dim A \leqslant 4$ then accept
6. For each $k \in \{2, \ldots, \max K\}$ do
   - $m_k := |\{k' \in K.\ k \text{ divides } k'\}|$
   - If $m_k - 1 > d$ then check the Jordan blocks for all primitive roots of unity of degree $k$
7. Accept

# Experiments

- input: $d = 0$ and matrix $A$ of dimension 21 with

$$\chi_A = \frac{4096x^{21} - 8192x^{20} + \ldots + 152x^6 - x^4 - 9x^3 + 1}{4096}$$

- basic certification algorithm
  - factor $\chi_A = \frac{(x+1)(x^2+1)(x^2+x+1)\cdot\left((x-1)(64x^7-64x^6+4x^3-1)\right)^2}{4096}$
  - compute norms of roots of $64x^7 - 64x^6 + 4x^3 - 1$
  - timeout after 1 hour
- new certification algorithm
  - apply Sturm's method
  - decompose $\chi_A = (x^3 - 1) \cdot (x^4 - 1) \cdot f$, $K = \{3, 4\}$
  - only check Jordan block of eigenvalue 1
  - finished within fraction of a second
- matrices of termComp (dim $A \leqslant 5$): new algorithm 5x faster

# Improvements in Automation

- new certification runs in polynomial time for $\dim A \leqslant 5$

$\Longrightarrow$ there exists polynomial time SAT/SMT-encoding

$\Longrightarrow$ possibility to encode desired degree when searching for matrix interpretation

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# Perron–Frobenius Theorem

- Parts 1 and 2 are well-studied
- New part 3: if $A$ is real non-negative matrix, then
  $\exists f\, K.\ \chi_A = f \cdot \prod_{k \in K}(x^k - \rho_A^k) \wedge (f(y) = 0 \longrightarrow |y| < \rho_A)$

## Proof by induction on the dimension of A.

- If $A$ is irreducible, then apply part 2 and set $K = \{k\}$
- If $\dim\ A = 1$ then result is trivial: $f = 1$, $K = \{1\}$
- Otherwise,

$$\pi(A) = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

  where $\pi$ is a permutation of rows and columns

$\Longrightarrow$ $B$ and $D$ are real non-negative matrices, $\chi_A = \chi_B \cdot \chi_D$

$\Longrightarrow$ apply induction hypothesis and perform case analysis:
$\rho_B = \rho_D \vee \rho_B < \rho_D \vee \rho_B > \rho_D$

# Largest Jordan blocks

Step 5 of new algorithm (if $\dim A \leqslant 4$ then accept) requires

## Theorem

*If $A$ is non-negative real matrix, $\dim A \leqslant 4$ and $\rho_A \leqslant 1$ then for every JB with $|\lambda| = 1$ there exists JB of 1 which is at least as large*

## Proof.

- Let there be JB with $\lambda \neq 1$, $|\lambda| = 1$ and size $s$ such that all JBs of 1 are smaller than $s$

$\Longrightarrow$ $s > 1$ since $\rho_A = 1$ is eigenvalue

$\Longrightarrow$ $\chi_A = (x^2 - 1)^2$

$$\Longrightarrow \pi(A) = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & a & c & d \\ \frac{1}{a} & 0 & e & f \\ 0 & 0 & 0 & b \\ 0 & 0 & \frac{1}{b} & 0 \end{pmatrix} \text{ for } \begin{array}{cc} a, b & > 0 \\ c, d, e, f & \geq 0 \end{array}$$

# Proof continued

- $\pi(A) = PJP^{-1}$ and $h = 0 \longrightarrow g = 0$ for

$$g = \frac{-abe + af + bc - d}{2b}$$

$$h = \frac{abe + af + bc + d}{2a}$$

$$J = \begin{pmatrix} -1 & g & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{2} & \frac{-a}{2} & \frac{abe+af-bc-d}{8b} & \frac{abe+af-bc-d}{8} \\ 0 & 0 & \frac{1}{2} & \frac{-b}{2} \\ \frac{1}{a} & 1 & \frac{abe-af+bc-d}{2ab} & 0 \\ 0 & 0 & \frac{1}{b} & 1 \end{pmatrix}$$

# Largest Jordan blocks

**Theorem**

*If $A$ is non-negative real matrix, $\dim A \leqslant 4$ and $\rho_A \leqslant 1$ then for every JB with $|\lambda| = 1$ there exists JB of 1 which is at least as large*

**Conjecture**

If $A$ is non-negative real matrix and $\rho_A \leqslant 1$ then for every JB with $|\lambda| = 1$ there exists JB of 1 which is at least as large

- with conjecture it would suffice to only consider JB for 1
- no violation of conjecture among billions of tested matrices
- no idea how to prove it

# Overview

# Overview on Formalization

- Carrier-based matrices ($\mathbb{N} \times \mathbb{N} \times (\mathbb{N} \to \mathbb{N} \to \alpha)$) for part 3
  - permits decomposition of matrices into smaller ones
- Type-based matrices ($\iota$ :: finite $\to \iota \to \alpha$) for part 1
  - continuity of matrix operations, Brouwer's fixpoint theorem
- Combination for part 2
  - proof requires continuity as well as decomposition
  - transfer, local type definitions
    (see paper or master seminar 2 in SS 2016)

# Summary

- efficient algorithm for certifying polynomial growth of $A^n$ for non-negative real matrices
- conjecture on largest Jordan blocks for further simplification
- permits SAT/SMT encoding for $\dim A \leqslant 5$
- soundness based on Perron–Frobenius theorem
- Isabelle formalization available in archive of formal proofs
- further application: unique solutions of Markov chains