Formalizing Completeness of Ordered Completion

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Master Seminar 1 @ CL

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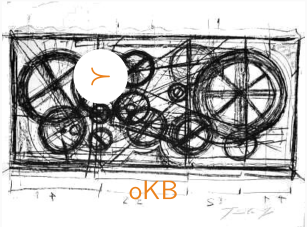
Ordered Completion



Ordered Completion

$$x \cdot 1 \approx x$$
$$x \cdot x^{-} \approx 1$$
$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$
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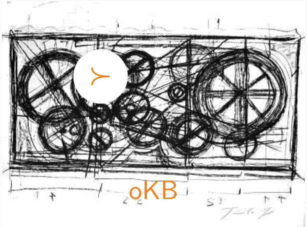
$\longrightarrow$



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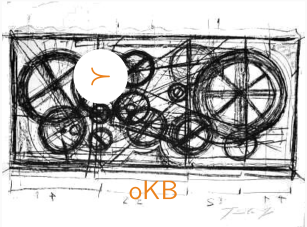

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ground complete presentation

$R \cup E$

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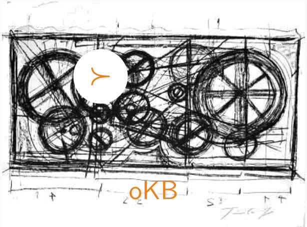

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- can decide ground equality:

$$(\mathsf{a} \cdot \mathsf{b}^{-})^{-} \approx \mathsf{b} \cdot \mathsf{a}^{-} \quad \text{because} \quad (\mathsf{a} \cdot \mathsf{b}^{-})^{-} \xrightarrow[R \cup E^{\succ}]{*} \cdot \xleftarrow[R \cup E^{\succ}]{*} \mathsf{b} \cdot \mathsf{a}^{-}$$

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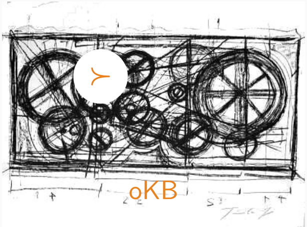

$\longrightarrow$

$$1^- \to 1$$
$$x \cdot 1 \to x$$
$$x \cdot x^- \to 1$$
$$(x^-)^- \to x$$
$$(x \cdot y)^- \to x^- \cdot y^-$$
$$(x \cdot y) \cdot z \to x \cdot (y \cdot z)$$
$$x \cdot y \approx y \cdot x$$

ground complete presentation

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- can decide ground equality:

$$(\mathsf{a} \cdot \mathsf{b}^-)^- \approx \mathsf{b} \cdot \mathsf{a}^- \quad \text{because} \quad (\mathsf{a} \cdot \mathsf{b}^-)^- \xrightarrow[R \cup E^{\succ}]{*} \cdot \xleftarrow[R \cup E^{\succ}]{*} \mathsf{b} \cdot \mathsf{a}^-$$

Correctness Theorem

Any fair oKB run produces ground complete presentation.

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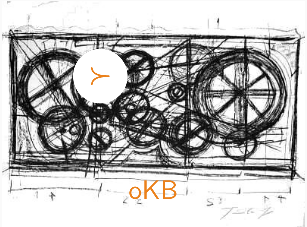

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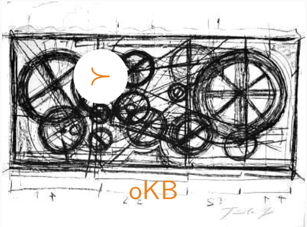

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complete presentation R

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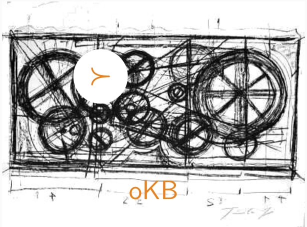

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complete presentation R

- can decide any equality:

$$(x \cdot y^{-})^{-} \approx y \cdot x^{-} \quad \text{because} \quad (x \cdot y^{-})^{-} \xrightarrow[R]{*} \cdot \xleftarrow[R]{*} y \cdot x^{-}$$

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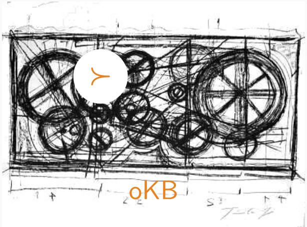

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Question (Completeness)

Under which circumstances will oKB compute a complete presentation?

Outline

- Ordered Completion
- Completeness Results
  - Ground Total Reduction Orders
  - Linear Systems
- Conclusion

Ordered Completion

Definitions

TRS R is

- terminating if there is no infinite sequence $t_0 \rightarrow_R t_1 \rightarrow_R t_2 \rightarrow_R \ldots$

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Example

$R = \{\mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{c}\}$ is

- complete ✓
- complete presentation of $\mathsf{a} \approx \mathsf{b}, \mathsf{b} \approx \mathsf{c}, \mathsf{c} \approx \mathsf{d}$ ✗

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Definitions

$t{\downarrow_R}$ denotes normal form of t with respect to R

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Consider set of equations E and reduction order $\succ$.

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Example

for $E = \{x + y \approx y + x\}$ and LPO with $\mathsf{a} > \mathsf{b} > \mathsf{0} > +$ have

$$E^{\succ} = \begin{cases} \mathsf{a} + \mathsf{b} \to \mathsf{b} + \mathsf{a} & (x + y) + x \to x + (x + y) & \dots \\ \mathsf{a} + \mathsf{0} \to \mathsf{0} + \mathsf{a} & \mathsf{a} + (\mathsf{b} + \mathsf{b}) \to (\mathsf{b} + \mathsf{b}) + \mathsf{a} & \end{cases}$$

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Definition (Extended Critical Pairs)

Let $\ell_1 \approx r_1$ and $\ell_2 \approx r_2 \in E^{\pm}$ such that

- $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_2)$
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$\ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma$

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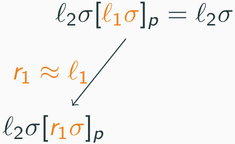

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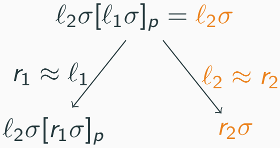

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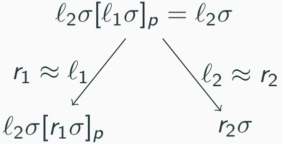

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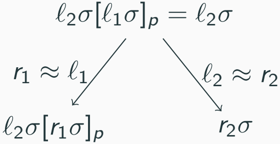

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Set of extended critical pairs among equations in E is denoted $\mathrm{CP}_{\succ}(E)$.

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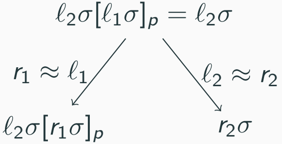

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Example

$1 \cdot (-x + x) \approx 0$ and $y + -y \approx -x + x$ give rise to $CP_{\succ}$ $1 \cdot (y + -y) \approx 0$:

$$1 \cdot (y + -y) \leftarrow 1 \cdot (-x + x) \to 0$$

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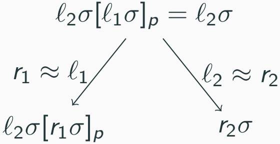

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Definition

reduction order $\succ$ is ground total if it is total on ground terms

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E: set of equations R: set of rewrite rules $\succ$: reduction order

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delete $$\frac{E \cup \{s \approx s\}, R}{E, R}$$

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E: set of equations $\qquad$ R: set of rewrite rules $\qquad$ $\succ$: reduction order

delete $\quad \dfrac{E \cup \{s \approx s\}, R}{E, R}$

orient $\quad \dfrac{E \cup \{s \approx t\}, R}{E, R \cup \{s \to t\}} \quad \dfrac{E \cup \{t \approx s\}, R}{E, R \cup \{s \to t\}} \quad$ if $s \succ t$

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compose $\quad \dfrac{E, R \cup \{s \to t\}}{E, R \cup \{s \to u\}} \quad$ if $t \to_{R \cup E^{\succ}} u$

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orient	$\dfrac{E \cup \{s \approx t\}, R}{E, R \cup \{s \to t\}}$	$\dfrac{E \cup \{t \approx s\}, R}{E, R \cup \{s \to t\}}$	if $s \succ t$
compose	$\dfrac{E, R \cup \{s \to t\}}{E, R \cup \{s \to u\}}$		if $t \to_{R \cup E^{\succ}} u$
simplify	$\dfrac{E \cup \{s \approx t\}, R}{E \cup \{s \approx u\}, R}$	$\dfrac{E \cup \{t \approx s\}, R}{E \cup \{u \approx s\}, R}$	if $t \xrightarrow{\cdot\!\!\triangleright_1}_{R \cup E^{\succ}} u$

Definition (Ordered Completion)

E: set of equations $\quad$ R: set of rewrite rules $\quad$ $\succ$: reduction order

delete $\quad \dfrac{E \cup \{s \approx s\}, R}{E, R}$

orient $\quad \dfrac{E \cup \{s \approx t\}, R}{E, R \cup \{s \to t\}} \quad \dfrac{E \cup \{t \approx s\}, R}{E, R \cup \{s \to t\}} \quad$ if $s \succ t$

compose $\quad \dfrac{E, R \cup \{s \to t\}}{E, R \cup \{s \to u\}} \quad$ if $t \to_{R \cup E^{\succ}} u$

simplify $\quad \dfrac{E \cup \{s \approx t\}, R}{E \cup \{s \approx u\}, R} \quad \dfrac{E \cup \{t \approx s\}, R}{E \cup \{u \approx s\}, R} \quad$ if $t \xrightarrow{\rhd_1}_{R \cup E^{\succ}} u$

collapse $\quad \dfrac{E, R \cup \{t \to s\}}{E \cup \{u \approx s\}, R} \quad$ if $t \xrightarrow{\rhd_2}_{R \cup E^{\succ}} u$

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E: set of equations $\qquad$ R: set of rewrite rules $\qquad$ $\succ$: reduction order

delete	$\dfrac{E \cup \{s \approx s\}, R}{E, R}$		
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simplify	$\dfrac{E \cup \{s \approx t\}, R}{E \cup \{s \approx u\}, R}$	$\dfrac{E \cup \{t \approx s\}, R}{E \cup \{u \approx s\}, R}$	if $t \xrightarrow{\rhd_1}_{R \cup E^{\succ}} u$
collapse	$\dfrac{E, R \cup \{t \to s\}}{E \cup \{u \approx s\}, R}$		if $t \xrightarrow{\rhd_2}_{R \cup E^{\succ}} u$
deduce	$\dfrac{E, R}{E \cup \{s \approx t\}, R}$		if $s \leftrightarrow_{R \cup E} \cdot \leftrightarrow_{R \cup E} t$

Example

$$1 \cdot (-x + x) \approx 0$$

$$1 \cdot (x + -x) \approx x + -x$$

$$-x + x \approx y + -y$$

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- LPO with precedence $+ > - > 0$

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- orient $1 \cdot (-x + x) > 0$

Example

$$1 \cdot (-x + x) \to 0$$

$$1 \cdot (x + -x) \approx x + -x$$

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- LPO with precedence $+ > - > 0$
- deduce $1 \cdot (y + -y) \leftarrow 1 \cdot (-x + x) \to 0$

Example

$$1 \cdot (y + -y) \approx 0 \qquad 1 \cdot (-x + x) \to 0$$

$$1 \cdot (x + -x) \approx x + -x$$

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Example

$$1 \cdot (-x + x) \to 0$$

$$0 \approx x + -x \qquad 1 \cdot (y + -y) \to 0$$

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$$0 \approx x + -x \qquad\qquad 1 \cdot (-x + x) \to 0$$

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Example

$$1 \cdot (-x + x) \to 0$$

$$1 \cdot (y + -y) \to 0$$

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- simplify $\quad y + -y \to 0$

Example

$$-x + x \approx 0$$

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$$1 \cdot (y + -y) \to 0$$

$$-x + x \approx 0 \qquad\qquad x + -x \to 0$$

- LPO with precedence $+ > - > 0$
- collapse $\quad y + -y \to 0$

Example

$$1 \cdot 0 \approx 0 \qquad 1 \cdot (-x + x) \to 0$$

$$-x + x \approx 0 \qquad x + -x \to 0$$

- LPO with precedence $+ > - > 0$

Example

$$1 \cdot 0 \approx 0 \qquad\qquad 1 \cdot (-x + x) \to 0$$

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- orient $\quad 1 \cdot 0 > 0$

Example

$$-x + x \approx 0$$

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- collapse $\quad -x + x \to 0$

Example

$$1 \cdot 0 \approx 0$$

$$\begin{aligned} 1 \cdot 0 &\to 0 \\ x + -x &\to 0 \\ -x + x &\to 0 \end{aligned}$$

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Example

$$1 \cdot 0 \approx 0$$

$$1 \cdot 0 \to 0$$

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- simplify $1 \cdot 0 \to 0$

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Definition

possibly infinite run

$$\gamma\colon\ (E_0, \varnothing) \vdash (E_1, R_1) \vdash (E_2, R_2) \vdash \cdots$$

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- $E_\infty = \bigcup_i E_i \qquad R_\infty = \bigcup_i R_i \qquad S_\infty = R_\infty \cup E_\infty^{\succ}$
- persistent equations and rules:
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Correctness Theorem

Bachmair, Dershowitz, and Plaisted '89

If γ is fair and $\succ$ is ground total then S_ω is ground complete presentation of E_0.

Definition

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proof based on proof orders:
compare conversions with $(\succ_{\mathsf{mul}}, \succ, \rhd, \succ_{\mathsf{mul}})_{\mathsf{lex}}$

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Infinite Runs in Abstract Completion.
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no proof orders, "separation of concerns"

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Completeness Results

This Section

$\mathcal{R}$ is canonical presentation of E_0 such that $\mathcal{R} \subseteq \succ$

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Definition

run $\gamma\colon (E_0, \varnothing) \vdash (E_1, R_1) \vdash (E_2, R_2) \vdash \cdots$ is simplifying if

- equations in E_ω are nontrivial and irreducible with respect to S_ω
- R_ω is reduced

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Completeness Theorem (1)

Bachmair, Dershowitz and Plaisted '89

Suppose γ uses ground total reduction order $\succ$, is simplifying, and satisfies $\mathrm{CP}_\succ(R_\omega \cup E_\omega) \subseteq E_\infty$.
Then $E_\omega = \varnothing$ and $R_\omega \doteq \mathcal{R}$.

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equal up to renaming variables

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Completeness Theorem (2) — Devie '90

Suppose E_0 is linear and γ is simplifying linear completion run using $\succ$ and satisfying $\mathrm{LCP}(R_\omega \cup E_\omega) \subseteq E_\infty$.
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proofs use (different) proof orders, and are rather monolithic

Completeness Theorem (2)

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The Case of a Ground Total Order

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Key Idea: Disguise Variables

- suppose E_0 is over terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$

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- write $\hat{t}$ for ground term obtained from t by replacing every variable x by $\hat{x}$
- have $s \to_R t$ iff $\hat{s} \to_R \hat{t}$ for every TRS R over $\mathcal{T}(\mathcal{F}, \mathcal{V})$

Roadmap to Completeness

Order Extension Lemma

There exists ground-total reduction order $\succ^{\mathcal{K}}$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$ such that $\succ \subseteq \succ^{\mathcal{K}}$

Roadmap to Completeness

Order Extension Lemma

There exists ground-total reduction order $\succ^{\mathcal{K}}$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$ such that $\succ \subseteq \succ^{\mathcal{K}}$

Proof.

- define $>_{\mathsf{kbo}}$ with $c > f$ for all $c \in \mathcal{K}$, $f \in \mathcal{F}$
- for $\bot$ minimal constant in $\mathcal{F}$, let $t_\bot$ be term where every variable replaced by $\bot$
- define $s \succ^{\mathcal{K}} t$ as $s_\bot \succ t_\bot$ or $s_\bot = t_\bot$ and $s >_{\mathsf{kbo}} t$

Roadmap to Completeness

Order Extension Lemma

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↓

Ground Completeness on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$

$S_\omega^{\mathcal{K}}$ is ground complete presentation of E_0 with respect to $\succ^{\mathcal{K}}$

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Peak Analysis Lemma

if $s \xleftarrow[r \approx \ell \in E_\omega{}^{\pm}]{\epsilon,\sigma} \cdot \xrightarrow[S_\omega]{} t$ and $r\sigma \not\succ \ell\sigma$ then $s \leftrightarrow_{E_\infty} t'$ for some $t \succeq t'$ or $s \notin \mathrm{NF}(S_\omega)$

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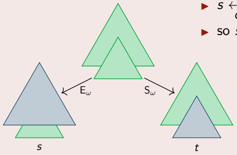
Proof.

(a) proper overlap



- $s \xleftrightarrow[CP_\succ(\mathsf{E}_\omega \cup R_\omega)]{} t$
- so $s \xleftrightarrow[\mathsf{E}_\infty]{} t$ by fairness

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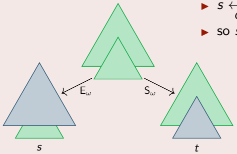

- $s \xleftrightarrow[CP_{\succ}(E_\omega \cup R_\omega)]{} t$
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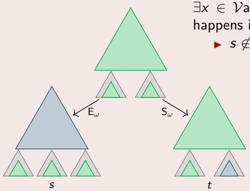
(b) variable overlap



$\exists x \in \mathcal{V}\mathrm{ar}(\ell)$ such that S_ω-step happens in $\sigma(x)$

- $s \notin \mathrm{NF}(S_\omega)$ if $x \in \mathcal{V}\mathrm{ar}(r)$

Roadmap to Completeness

Order Extension Lemma

There exists ground-total reduction order $\succ^{\mathcal{K}}$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$ such that $\succ \subseteq \succ^{\mathcal{K}}$

↓

Ground Completeness on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$

$S_\omega^{\mathcal{K}}$ is ground complete presentation of E_0 with respect to $\succ^{\mathcal{K}}$

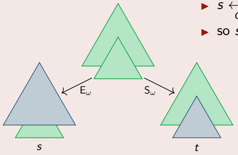
↓

Peak Analysis Lemma

if $s \xleftarrow[r \approx \ell \in E_\omega^{\pm}]{\epsilon, \sigma} \cdot \xrightarrow[S_\omega]{} t$ and $r\sigma \not\succ \ell\sigma$ then $s \leftrightarrow_{E_\infty} t'$ for some $t \succeq t'$ or $s \notin \mathrm{NF}(S_\omega)$

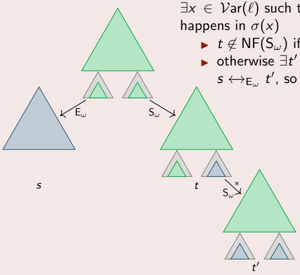
Proof.

(a) proper overlap



- $s \xleftrightarrow[CP_\succ(E_\omega \cup R_\omega)]{} t$
- so $s \xleftrightarrow[E_\infty]{} t$ by fairness

(b) variable overlap



$\exists x \in \mathcal{V}\mathrm{ar}(\ell)$ such that S_ω-step happens in $\sigma(x)$

- $t \notin \mathrm{NF}(S_\omega)$ if $x \in \mathcal{V}\mathrm{ar}(r)$
- otherwise $\exists t' \preceq t$ with $s \leftrightarrow_{E_\omega} t'$, so $s \leftrightarrow_{E_\infty} t'$

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Normal Form Persistence

$\mathsf{NF}(S_\infty) = \mathsf{NF}(S_\omega)$

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-Lemma

if $\hat{s} \rightarrow_{S_\infty^{\mathcal{K}}} \hat{t}$ then $s \notin \mathsf{NF}(S_\omega)$

Norr
NF(

Proof.
Induction on $\hat{t}$ with respect to $\succ^{\mathcal{K}}$.
$\exists \ell \approx r \in E_\infty{}^{\pm} \cup R_\infty$ such that

$$\hat{s} = \hat{C}[\ell\hat{\sigma}] \xrightarrow[S_\infty^{\mathcal{K}}]{} \hat{C}[r\hat{\sigma}] = \hat{t}$$

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NF(…

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$\exists \ell \approx r \in {E_\infty}^{\pm} \cup R_\infty$ such that

$$\hat{s} = \hat{C}[\ell\hat{\sigma}] \xrightarrow[S_\infty^{\mathcal{K}}]{} \hat{C}[r\hat{\sigma}] = \hat{t}$$

$\hat{C} \neq \square$:

- $\ell\hat{\sigma} \to_{S_\infty^{\mathcal{K}}} r\hat{\sigma}$
- have $\hat{t} \succ^{\mathcal{K}} r\hat{\sigma}$
- conclude $\ell\sigma \notin \mathrm{NF}(S_\infty)$ by IH

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$\hat{C} = \square$:

(1) if $s \succ t$ then $s \to_{S_\infty} t$, so $s \notin \mathrm{NF}(S_\infty) = \mathrm{NF}(S_\omega)$ ✓

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There exists ground-total reduction order $\succ^{\mathcal{K}}$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$ such that $\succ \subseteq \succ^{\mathcal{K}}$

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Norr…

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(1) if $s \succ t$ then $s \rightarrow_{S_\infty} t$, so $s \notin \mathrm{NF}(S_\infty) = \mathrm{NF}(S_\omega)$ ✓

(2) $\ell \approx r \in E_\infty \setminus E_\omega$, so $\ell \approx r \in E_i \setminus E_{i+1}$

- $\ell = r$ or $r \rightarrow \ell \in R_{i+1}$
  impossible as $\ell\hat{\sigma} \succ r\hat{\sigma}$

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- $\ell \rightarrow r \in R_{i+1}$ or $\ell \rightarrow_{S_{i+1}} \cdot E_{i+1}\ r$
  ℓ is S_∞-reducible, hence S_ω-reducible

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-Lemma

if $\hat{s} \to_{S_\infty^{\mathcal{K}}} \hat{t}$ then $s \notin \mathrm{NF}(S_\omega)$

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$\hat{C} = \Box$:

(1) if $s \succ t$ then $s \to_{S_\infty} t$, so $s \notin \mathrm{NF}(S_\infty) = \mathrm{NF}(S_\omega)$ ✓

(2) $\ell \approx r \in E_\infty \setminus E_\omega$, so $\ell \approx r \in E_i \setminus E_{i+1}$ ✓

- $\ell = r$ or $r \to \ell \in R_{i+1}$
  impossible as $\ell\hat{\sigma} \succ r\hat{\sigma}$
- $\ell \to r \in R_{i+1}$ or $\ell \to_{S_{i+1}} \cdot E_{i+1}\ r$
  ℓ is S_∞-reducible, hence S_ω-reducible
- $\ell\ E_{i+1} \cdot {}_{S_{i+1}}\!\leftarrow r$
  conclude by IH

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$\hat{C} = \square$:

(1) if $s \succ t$ then $s \to_{S_\infty} t$, so $s \notin \mathrm{NF}(S_\infty) = \mathrm{NF}(S_\omega)$ ✓

(2) $\ell \approx r \in E_\infty \setminus E_\omega$, so $\ell \approx r \in E_i \setminus E_{i+1}$ ✓

(3) $\ell \approx r \in E_\omega$

- have $t \notin \mathrm{NF}(\mathcal{R})$, so $\hat{t} \notin \mathrm{NF}(S_\infty)$
- thus $\exists$ step $\hat{t} \to_{S_\infty^{\mathcal{K}}} \hat{u}$ so $t \notin \mathrm{NF}(S_\omega)$ by IH
- peak analysis of $s \xleftarrow[\ell \approx r]{} t \xrightarrow[S_\omega]{} v$
  - if $s \notin \mathrm{NF}(S_\omega)$ we are done
  - if $s \leftrightarrow_{E_\infty} v'$ for some $v' \preceq v$ then $\hat{s} \to_{S_\infty^{\mathcal{K}}} \hat{v}'$ with $\hat{t} \succ^{\mathcal{K}} \hat{v}'$, conclude by IH

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-Lemma

if $\hat{s} \rightarrow_{S_\infty^{\mathcal{K}}} \hat{t}$ then $s \notin \mathsf{NF}(S_\omega)$

Normal Form Lemma

$\mathsf{NF}(S_\omega) \subseteq \mathsf{NF}(\mathcal{R})$

Normal Form Persistence

$\mathsf{NF}(S_\infty) = \mathsf{NF}(S_\omega)$

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Normal Form Persistence

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-Lemma

if $\hat{s} \to_{S_\infty^{\mathcal{K}}} \hat{t}$ then $s \notin \mathsf{NF}(S_\omega)$

Normal Form Lemma

$\mathsf{NF}(S_\omega) \subseteq \mathsf{NF}(\mathcal{R})$

Proof.

- if $t \to_{\mathcal{R}} u$ then $t \succ u$
- by ground completeness $\hat{t} \downarrow_{S_\omega^{\mathcal{K}}} \hat{u}$ and $\hat{t} \succ^{\mathcal{K}} \hat{u}$
- $\hat{t}$ must be $S_\omega^{\mathcal{K}}$-reducible, hence $S_\infty^{\mathcal{K}}$-reducible
- conclude by Key Lemma

Roadmap to Completeness

Order Extension Lemma

There exists ground-total reduction order $\succ^{\mathcal{K}}$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$ such that $\succ \subseteq \succ^{\mathcal{K}}$

↓

Ground Completeness on $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$

$S_\omega^{\mathcal{K}}$ is ground complete presentation of E_0 with respect to $\succ^{\mathcal{K}}$

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-Lemma

if $\hat{s} \to_{S_\infty^{\mathcal{K}}} \hat{t}$ then $s \notin \mathsf{NF}(S_\omega)$

↓

Normal Form Lemma

$\mathsf{NF}(S_\omega) \subseteq \mathsf{NF}(\mathcal{R})$

↓

Completeness Theorem

if (E_ω, R_ω) is simplified then $E_\omega = \varnothing$ and R_ω is complete

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$\mathsf{NF}(S_\infty) = \mathsf{NF}(S_\omega)$

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if $\hat{s} \to_{S_\infty^{\mathcal{K}}} \hat{t}$ then $s \notin \mathsf{NF}(S_\omega)$

Normal Form Persistence

$\mathsf{NF}(S_\infty) = \mathsf{NF}(S_\omega)$

Normal Form Lemma

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Completeness Theorem

if (E_ω, R_ω) is simplified then $E_\omega = \varnothing$ and R_ω is complete

Proof.

- E_ω must be $\mathcal{R}$-reducible by completeness of $\mathcal{R}$
- by Normal Form Lemma E_ω must be S_ω-reducible
- as E_ω is simplified must have $E_\omega = \varnothing$
- from Normal Form Lemma and $R_\omega \subseteq \mathcal{R}$ conclude $R_\omega \doteq \mathcal{R}$ by results on normalization equivalence

The Linear Case

Example (Devie 1990)

$$
\begin{aligned}
&\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_2(x))))) && \mathsf{h}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(x)) && \mathsf{f}_1(\mathsf{a}) \approx \mathsf{a} \\
&\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_1(x))))) && \mathsf{h}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(x)) && \mathsf{f}_2(\mathsf{a}) \approx \mathsf{a} \\
&\mathsf{g}_1(\mathsf{a}) \approx \mathsf{a} && \mathsf{h}_1(\mathsf{a}) \approx \mathsf{a} && \mathsf{i}_1(\mathsf{a}) \approx \mathsf{a} \\
&\mathsf{g}_2(\mathsf{a}) \approx \mathsf{a} && \mathsf{h}_2(\mathsf{a}) \approx \mathsf{a} && \mathsf{i}_2(\mathsf{a}) \approx \mathsf{a}
\end{aligned}
$$

The Linear Case

Example (Devie 1990)

$$
\begin{array}{lll}
f_1(g_1(i_1(x))) \approx g_1(i_1(f_1(g_1(i_2(x))))) & h_1(g_1(i_1(x))) \approx g_1(i_1(x)) & f_1(a) \approx a \\
f_2(g_2(i_2(x))) \approx g_2(i_2(f_2(g_2(i_1(x))))) & h_2(g_2(i_2(x))) \approx g_2(i_2(x)) & f_2(a) \approx a \\
g_1(a) \approx a & h_1(a) \approx a & i_1(a) \approx a \\
g_2(a) \approx a & h_2(a) \approx a & i_2(a) \approx a
\end{array}
$$

- orienting all equations from left to right yields a canonical system R

The Linear Case

Example (Devie 1990)

$$
\begin{array}{ccc}
\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_2(x))))) & \mathsf{h}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(x)) & \mathsf{f}_1(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_1(x))))) & \mathsf{h}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(x)) & \mathsf{f}_2(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{g}_1(\mathsf{a}) \approx \mathsf{a} & \mathsf{h}_1(\mathsf{a}) \approx \mathsf{a} & \mathsf{i}_1(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{g}_2(\mathsf{a}) \approx \mathsf{a} & \mathsf{h}_2(\mathsf{a}) \approx \mathsf{a} & \mathsf{i}_2(\mathsf{a}) \approx \mathsf{a}
\end{array}
$$

- orienting all equations from left to right yields a canonical system R
- cannot extend $\rightarrow_R^+$ to ground total reduction order: $\mathsf{i}_1(\mathsf{a}) \succ \mathsf{i}_2(\mathsf{a})$ implies $\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(\mathsf{a}))) \succ \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_1(\mathsf{a}))))) \succ \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(\mathsf{a}))))) \succ \mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(\mathsf{a})))$

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$$
\begin{array}{rll}
\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(\mathsf{f}_1(\mathsf{g}_1(\mathsf{i}_2(x))))) & \mathsf{h}_1(\mathsf{g}_1(\mathsf{i}_1(x))) \approx \mathsf{g}_1(\mathsf{i}_1(x)) & \mathsf{f}_1(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_1(x))))) & \mathsf{h}_2(\mathsf{g}_2(\mathsf{i}_2(x))) \approx \mathsf{g}_2(\mathsf{i}_2(x)) & \mathsf{f}_2(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{g}_1(\mathsf{a}) \approx \mathsf{a} & \mathsf{h}_1(\mathsf{a}) \approx \mathsf{a} & \mathsf{i}_1(\mathsf{a}) \approx \mathsf{a} \\
\mathsf{g}_2(\mathsf{a}) \approx \mathsf{a} & \mathsf{h}_2(\mathsf{a}) \approx \mathsf{a} & \mathsf{i}_2(\mathsf{a}) \approx \mathsf{a}
\end{array}
$$

- orienting all equations from left to right yields a canonical system R
- cannot extend $\to_R^+$ to ground total reduction order: $\mathsf{i}_1(\mathsf{a}) \succ \mathsf{i}_2(\mathsf{a})$ implies $\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(\mathsf{a}))) \succ \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_1(\mathsf{a}))))) \succ \mathsf{g}_2(\mathsf{i}_2(\mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(\mathsf{a}))))) \succ \mathsf{f}_2(\mathsf{g}_2(\mathsf{i}_2(\mathsf{a})))$
- ordered completion cannot produce a finite complete system when using ground-total reduction order

Definition (Linear Ordered Completion $\vdash_{\mathsf{lin}}$)

E: set of equations R: set of rewrite rules $\succ$: reduction order

delete	$\dfrac{E \cup \{s \approx s\}, R}{E, R}$		
orient	$\dfrac{E \cup \{s \approx t\}, R}{E, R \cup \{s \to t\}}$	$\dfrac{E \cup \{t \approx s\}, R}{E, R \cup \{s \to t\}}$	if $s \succ t$
compose	$\dfrac{E, R \cup \{s \to t\}}{E, R \cup \{s \to u\}}$		if $t \to_R u$
simplify	$\dfrac{E \cup \{s \approx t\}, R}{E \cup \{s \approx u\}, R}$	$\dfrac{E \cup \{t \approx s\}, R}{E \cup \{u \approx s\}, R}$	if $t \to_R u$
collapse	$\dfrac{E, R \cup \{t \to s\}}{E \cup \{u \approx s\}, R}$		if $t \xrightarrow{\rhd}_R u$
deduce	$\dfrac{E, R}{E \cup \{s \approx t\}, R}$		if $s \leftarrow_{R \cup E^{\pm}} u \to_{R \cup E^{\pm}} t$ and $s \approx t$ is linear

suppose (E_ω, R_ω) is result of run

$$\gamma\colon\ (E_0, \varnothing) \vdash_{\mathsf{lin}} (E_1, R_1) \vdash_{\mathsf{lin}} (E_2, R_2) \vdash_{\mathsf{lin}} \cdots$$

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Completeness Theorem (2) **Devie '90**

Suppose E_0 is linear and γ is simplifying linear completion run using $\succ$ and satisfying $\mathsf{LCP}(R_\omega \cup E_\omega) \subseteq E_\infty$.
Then $E_\omega = \varnothing$ and $R_\omega \doteq \mathcal{R}$.

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Definition (Linear Critical Pairs)

Let $\ell_1 \approx r_1$ and $\ell_2 \approx r_2$ in $E^\pm$ such that

- $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(\ell_2)$,
- $\sigma = \mathsf{mgu}(\ell_1, \ell_2|_p)$,
- $\ell_1 \succ r_1$ and $r_2 \not\succ \ell_2$, or $\ell_2 \succ r_2$ and $r_1 \not\succ \ell_1$

Then $\ell_2\sigma[r_2\sigma] \approx r_2\sigma$ is linear critical pair.
$\mathsf{LCP}(E)$ is set of all linear critical pairs among rules in E.

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Roadmap to Completeness

It's Just Ordered Completion-Lemma

$(E, R) \vdash^*_{\text{lin}} (E', R') \Longrightarrow (E, R) \vdash^* (E', R')$

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↓

E_∞-Lemma

$E_\infty \subseteq \rightarrow^*_{R_\infty} \cdot E^{=}_{\omega} \cdot {}_{R_\infty}{}^{*}\!\!\leftarrow$

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↓

Peak Analysis Lemma

$\xleftarrow[E_\omega^{\pm} \cup R_\omega]{r \leftarrow \ell} \cdot \xrightarrow[R_\omega]{} \subseteq \xrightarrow[R_\infty]{*} \cdot E_\omega^{=} \cdot \xleftarrow[R_\infty]{*}$ if $r \not\succ \ell$

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Linearity Preservation

If $(E,R) \vdash^* (E',R')$ and $E \cup R$ is linear then $E' \cup R'$ is linear

R_∞-Lemma

if $\ell \to r \in R_\infty$ then $\ell \to_{R_\omega} \cdot (\leftrightarrow^{\prec \ell}_{E_\omega \cup R_\omega})^*\ r$

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$(E, R) \vdash^*_{\mathsf{lin}} (E', R') \implies (E, R) \vdash^* (E', R')$

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$\mathsf{NF}(R_\omega) \subseteq \mathsf{NF}(\mathcal{R})$

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-Lemma

if $s \rightarrow^!_{\mathcal{R}} t$ then $s \rightarrow^*_{R_\omega} t$

Normal Form Lemma

$\mathrm{NF}(R_\omega) \subseteq \mathrm{NF}(\mathcal{R})$

Completeness Theorem

if (E_ω, R_ω) is simplified then $E_\omega = \varnothing$ and R_ω is complete

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If $(E, R) \vdash^* (E', R')$ and $E \cup R$ is linear then $E' \cup R'$ is linear

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if $\ell \rightarrow r \in R_\infty$ then $\ell \rightarrow_{R_\omega} \cdot (\leftrightarrow^{\prec \ell}_{E_\omega \cup R_\omega})^* \; r$

Conclusion

Formalization

Ground Total Order Case

- skolemization done via new type (no assumptions on given signature)

```
datatype ('a,'b) f_ext = FOrig 'a | FFresh 'b
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- 2200 LoC, 1500 thereof for lifting run to $\succ^{\mathcal{K}}$
- supports prime critical pair criterion

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Linear Case

- 1000 LoC
- could reuse general correctness result in linear case
  (original proofs used different proof orders)

Conclusion

Summary

- new proofs for two completeness results from literature: no proof orders, less monolithic—more formalization friendly?
- first formalization of completeness of ordered completion
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Open Problem

Suppose $\mathcal{R}$ is complete presentation of E_0 such that

- $\to_{\mathcal{R}}^{+}$ cannot be extended to ground total reduction order, and
- E_0 is not linear

Will ordered completion run using $\to_{\mathcal{R}}^{+}$ as reduction order find complete system?