

Normalising Strategies for Orthogonal Systems and Beyond

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Master Seminar 1, Wednesday October 18th, 2017



Problem

Setting

term rewrite systems representing arbitrary partial functions

- normal form represents result

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- at most one result

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Example

- combinatory logic (Schönfinkel, Curry), λ -calculus (Church)
- PCF (Scott, Plotkin, Milner)
- Haskell

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term rewrite systems representing arbitrary partial functions

- normal form represents result
- at most one result
- termination not decidable

Example

- combinatory logic (Schönfinkel, Curry), λ -calculus (Church)
- PCF (Scott, Plotkin, Milner)
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Question

strategy that always computes result, if that exists?

Strategy example

term:

$$f(1)$$

Strategy example

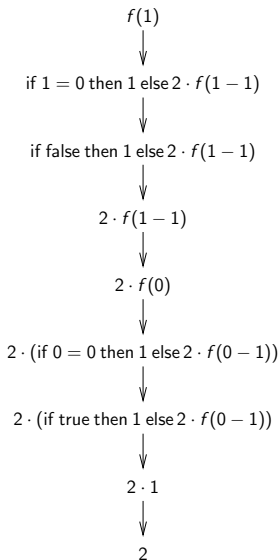
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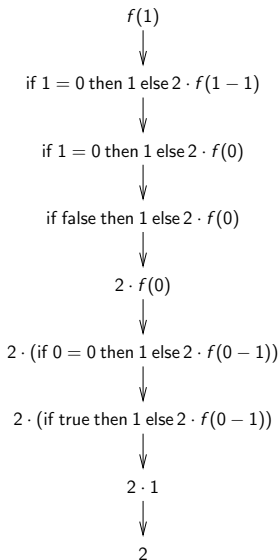
rewrite rules:

$$\begin{aligned}
 f(x) &\rightarrow \text{if } x = 0 \text{ then } 1 \text{ else } 2 \cdot f(x - 1) \\
 \text{if false then } x \text{ else } y &\rightarrow y \\
 \text{if true then } x \text{ else } y &\rightarrow x \\
 &\vdots
 \end{aligned}$$

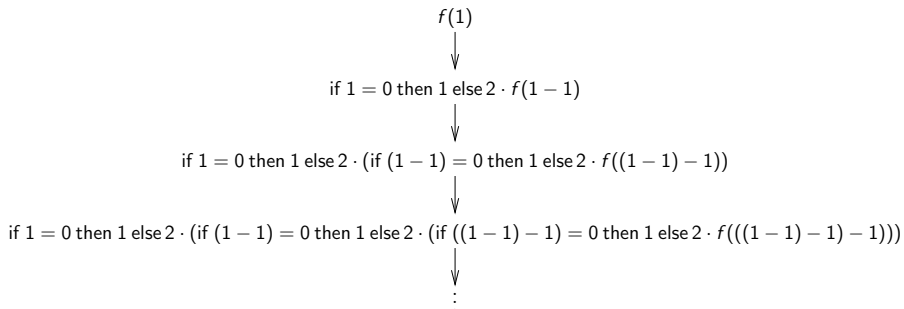
Strategy example



Strategy example



And Yet Another



Strategy

Definition

strategy is sub-system having same set of normal forms

Strategy

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Example

innermost reduction is a strategy

non-innermost reduction is **not** a strategy; lone redex is stuck

call-by-value is **not** a strategy; $(\lambda x.x)(z\lambda x.x)$ is stuck

Strategy

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strategy is sub-system having same set of normal forms

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innermost reduction is a strategy

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strategy is **normalising** if terminating on terms having normal form

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Example

normal order is normalising for left-normal systems (λ -calculus/CL)

innermost reduction is **not** normalising; K/Ω

Normalisation by Random Descent

VvO and Toyama, FSCD 2016

hyper-normalisation results for **left-normal** (CL, λ) systems:

Future work

- ▶ Retrofit known (hyper-)normalisation results in setting (may require slight generalisation of conversion monoid)
- ▶ Extend to other sets of normal forms (head, weak head)

This talk: retrofitting to orthogonal systems

Definition

term rewrite system is **orthogonal** if non-overlapping and left-linear

This talk: retrofitting to orthogonal systems

Definition

term rewrite system is **orthogonal** if non-overlapping and left-linear

Example

$$\begin{aligned} @ (I, x) &\rightarrow x \\ @ (@ (K, x), y) &\rightarrow x \\ @ (@ (@ (S, x), y), z) &\rightarrow @ (@ (x, z), @ (y, z)) \\ f(x, a, b) &\rightarrow c \\ f(b, x, a) &\rightarrow c \\ f(a, b, x) &\rightarrow c \end{aligned}$$

This talk: retrofitting to orthogonal systems

Definition

term rewrite system is **orthogonal** if non-overlapping and left-linear

Example

$$Ix \rightarrow x$$

$$Kxy \rightarrow x$$

$$Sxyz \rightarrow xz(yz)$$

$$f(x, a, b) \rightarrow c$$

$$f(b, x, a) \rightarrow c$$

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$a \rightarrow b$, $a \rightarrow c$ is **not** non-overlapping

$e(x, x) \rightarrow \top$, $e(x, f(x)) \rightarrow \perp$, $a \rightarrow f(a)$ is **not** left-linear

This talk: retrofitting to orthogonal systems

Definition

term rewrite system is **orthogonal** if non-overlapping and left-linear

Theorem (Rosen 73)

term in orthogonal rewrite system has at most one normal form

Proof.

by confluence/Church–Rosser property: any peak $s \leftarrow t \rightarrow u$ can be completed into valley $s \twoheadrightarrow r \leftarrow u$ by contracting **residuals**

$$S_{xyz} \leftarrow S_{xy}(l/z) \rightarrow x(l/z)(y(l/z))$$

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$$S_{xyz} \rightarrow xz(yz) \leftarrow x(l/z)(yz) \leftarrow x(l/z)(y(l/z))$$

colours indicate **descendants**; residual if redex-pattern descends \square

Normalising strategies for orthogonal systems?

Non-trivial

normal order strategy **not** normalising for orthogonal systems

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Example

normal order reduces $f(\Omega, la, b)$ to itself in union of CL and

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$$Sxyz \rightarrow xz(yz)$$

$$f(x, a, b) \rightarrow c$$

$$f(b, x, a) \rightarrow c$$

$$f(a, b, x) \rightarrow c$$

$$\Omega = SII(SII)$$

Needed

Definition (Huet, Lévy 91)

redex **needed** if residual contracted in any reduction to normal form

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Existence?

la is needed in $f(\Omega, la, b)$

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needed redex in $f(t_1, t_2, t_3)$? (assuming t_i not in normal form)

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$$g(a, x) \rightarrow a$$

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not orthogonal (but confluent)

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Normalisation?

$$f(\Omega, la, b) \rightarrow f(\Omega, a, b) \rightarrow c$$

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$f(\Omega, la, b) \rightarrow f(\Omega, a, b) \rightarrow c$

needed reduction may loop on a for

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Theorem (Huet, Lévy 91)

needed reduction is normalising strategy

Proof. (Huet, Lévy 91).

external strategy is needed and normalising
needed reduction bounded by external reduction □

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Proof. (Bethke, Klop, de Vrijer 99).

symbol **needed** if **contributes** to normal form

- all symbols in normal form needed
- $t \rightarrow s$, needed prefix s' of s has needed prefix t' of t **origin** \square

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$f(\Omega, la, b) \rightarrow f(\Omega, a, b) \rightarrow c$

External strategy

Idea

- non-needed if erased; $f(a)$ if $f(x) \rightarrow b$

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- **never** below a redex-pattern then definitely needed; e.g. head

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Example

in term $f(a, a)$ for rule

$$\begin{aligned} a &\rightarrow b \\ f(b, x) &\rightarrow c \end{aligned}$$

second a **not** below redex-pattern, but residual after one step **is**:
 $f(a, a) \rightarrow f(b, a) \rightarrow c$

External strategy

Definition

symbol **outer** if not below redex-pattern

redex-pattern **outer** if its head symbol is

symbol **external** if outer and descendant (if any) external again

redex-pattern **external** if outer and residual (if any) external again

External strategy

Definition

redex-pattern **external** if outer and residual (if any) external again

Lemma

*may witness non-externality by reduction at **parallel** positions*

Proof.

reduction R witnessing non-externality of f occurring in t

1. if step above f truncate before 1st such in R ; if redex-pattern
 - overlaps f then no descendant after in R ; contradiction
 - above f then **shorter** witness to non-externality
2. if step strictly below f drop last such in R ; **shorter** witness □

External strategy

Definition

redex-pattern **external** if outer and residual (if any) external again

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Definition

if $t = C[\ell^\sigma]_p \rightarrow C[r^\sigma]_p = s$ then **external origin** of external prefix s' of s is: if p in s' then $s'[\ell]_p$, otherwise s'

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Theorem

external origin of external prefix is external; contains external redex

Proof.

by lemma and orthogonality (for $s'[\ell]_p$) □

External strategy

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Definition

external strategy: if $t = f(\vec{t})$ head normal form then recurse on \vec{t} , otherwise $t \twoheadrightarrow \ell^\sigma$ and contract step in external origin of ℓ

External normalisation

Theorem

external reduction has random descent

Proof.

if $s \xrightarrow{e} t \rightarrow_e u$ then $s = u$, or $s \rightarrow_e r \xrightarrow{e} u$ □

External normalisation

Theorem

external reduction has random descent

Definition

distance $d(t)$ #external steps to normal form, else ∞

External normalisation

Theorem

external reduction has random descent

Definition

distance $d(t)$ #external steps to normal form, else ∞

Lemma

if $t \rightarrow s$ then $d(t) \geq d(s)$

Proof.

by induction on $d(t)$; interesting case t has but is not normal form:

$s \xrightarrow{e} t \rightarrow u$ and $s \rightarrow r \xleftarrow{e} u$ (by orthogonality)

$d(t) = d(s) + 1$, $d(s) \geq d(r)$ by IH, $d(r) + 1 = d(u)$ □

External normalisation

Theorem

external reduction has random descent

Definition

distance $d(t)$ #external steps to normal form, else ∞

Lemma

if $t \rightarrow s$ then $d(t) \geq d(s)$

Theorem

*external strategy is (**hyper-**)normalising*

Proof.

by lemma; if $t \rightarrow_e s$ then $d(t) > d(s)$ if t has normal form □

Bounding needed by external reductions

Lemma

if $t \rightarrow_n s$ then $d(t) > d(s)$ if t has normal form

Proof.

by induction on $d(t)$; interesting case t has but is not normal form:

$s \xrightarrow{e} t \rightarrow_n u$ and $s \twoheadrightarrow s' \rightarrow_n r' \twoheadrightarrow r \xrightarrow{e} u$

$d(t) = d(s) + 1$, $d(s) \geq d(s') > d(r') \geq d(r)$ by IH,

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Bounding needed by external reductions

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$d(r) + 1 = d(u)$ □

Example

external $>$ needed; term $f(a)$, rules $a \rightarrow b$, $f(x) \rightarrow g(x, x)$; $3 > 2$

Bounding needed by external reductions

Lemma

if $t \rightarrow_n s$ then $d(t) > d(s)$ if t has normal form

Proof.

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$d(r) + 1 = d(u)$ □

Theorem

needed strategy is (*hyper-*)normalising

Proof.

by lemmata distance never increases, decreases by needed steps □

Needed prefix vs. external prefix

Definition

if $t = C[\ell^\sigma]_p \rightarrow C[r^\sigma]_p = s$ then **external origin** of external prefix s' of s is: if p in s' then $s'[\ell]_p$, otherwise s'

Needed prefix vs. external prefix

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Definition (Bethke, Klop, de Vrijer 99)

if $t = C[\ell^\sigma]_p \rightarrow C[r^\sigma]_p = s$ then **needed origin** of needed prefix s' of s is: if p in s' then $s'[\ell^\tau]_p$ with τ **mgu** of $s'|_p$ and r , otherwise s'

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Example

$f(f(a)) \rightarrow f(a) \rightarrow a$ for rule $f(x) \rightarrow x$

external origin: $f(f(a)) \rightarrow f(a) \rightarrow a$

needed origin: $f(f(a)) \rightarrow f(a) \rightarrow a$

Normalisation by head needness

Definition

term t in **head** normal form if not $t \rightarrow \ell^\sigma$

head needed: residual contracted in reductions to head normal form

Normalisation by head needness

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term t in **head** normal form if not $t \rightarrow \ell^\sigma$

head needed: residual contracted in reductions to head normal form

Example

$f(a)$ in head normal form for $a \rightarrow b$, **not** for $a \rightarrow b, f(b) \rightarrow f(b)$

Normalisation by head needness

Definition

term t in **head** normal form if not $t \rightarrow \ell^\sigma$

head needed: residual contracted in reductions to head normal form

Definition

head external reduction: if $t \rightarrow \ell^\sigma$ contract step in external origin ℓ

Theorem

head needed reduction is (hyper-)head-normalising strategy

Proof.

head distance $hd(t)$ #head external steps to hnf

if $t \rightarrow s$ then $hd(t) \geq hd(s)$; if $t \rightarrow_{hn} s$ then $hd(t) > hd(s)$ \square

Non-orthogonal left-linear systems

Example

$\lambda\beta\eta$ -calculus; $\lambda x. I(KIx)x$ **no** external redex

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Idea

- random descent only requires redexes have **unique** residuals

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- so include redexes below **linear** (in rhs) variables

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Idea

- random descent only requires redexes have **unique** residuals
- so include redexes below **linear** (in rhs) variables
-

$$(\lambda x. M(x))N \rightarrow M(N)$$

$$\lambda x. Mx \rightarrow M$$

both linear in function (left argument @), function body (M)

Non-orthogonal left-linear systems

Example

$\lambda\beta\eta$ -calculus; $\lambda x. I(KIx)x$ **no** external redex

Definition

spine: if head normal form recur, else **head** spine

head spine: recur on left

Non-orthogonal left-linear systems

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$\lambda\beta\eta$ -calculus; $\lambda x. I(KIx)x$ **no** external redex

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Example

$\lambda x. I(KIx)x \rightarrow \lambda x. KIxx \rightarrow \lambda x. Ix \rightarrow I$

Non-orthogonal left-linear systems

Example

$\lambda\beta\eta$ -calculus; $\lambda x.I(KIx)x$ **no** external redex

Definition

spine: if head normal form recur, else **head** spine

head spine: recur on left

Example

$\lambda x.I(KIx)x \rightarrow \lambda x.KIx \rightarrow \lambda x.Ix \rightarrow I$

Theorem

spine reduction is (hyper-)normalising strategy

Non-orthogonal left-linear systems

Example

$\lambda\beta\eta$ -calculus; $\lambda x. I(KIx)x$ **no** external redex

Definition

spine: if head normal form recur, else **head** spine

head spine: recur on left

Example

$\lambda x. I(KIx)x \rightarrow \lambda x. KIxx \rightarrow \lambda x. Ix \rightarrow I$

Theorem

spine reduction is (hyper-)normalising strategy

Proof.

spine distance $ds(t)$ #spine steps to nf □

Conclusion

- (head) external strategy having random descent \Rightarrow distance
- distance \Rightarrow prove normalisation by induction on distance
- generalises from external prefixes to linear prefixes (non-ortho)