universität innsbruck

Seminar Report

Gödel's Incompleteness Theorems

Ahmet Aspir
Mark Nardi

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Supervisor: Dr. Georg Moser

Abstract

Gödel's incompleteness theorems are very fundamental for mathematics and computational logic. To put it in a nutshell, it focuses on the fact that certain true statements in a system, cannot be proven within the same system. In this paper, these theorems will be addressed and explained using an example. Also, incompleteness will be shown for specific arithmetic systems one of them being Peano Arithmetic with exponentiation.

1 Introduction

Formal systems are used in every aspect of science. Even our every-day math consists of such a formal system. Everything makes perfect sense and if we can find something new, we can be certain to be able to prove or disprove it, or do we? The Austrian mathematician and logician Gödel proved that this is indeed not the case. First, we will introduce some constructs Gödel defined, e.g. the Diagonalization, so that we can prove that there exist not-provable or not-disprovable sentences. Next, we will define the language we are using and some more tools for the proof. Lastly, we will show the Incompleteness of the Peano Arithmetic, either with Exponentiation and without it.

Note: All definitions, theorems, and corollaries are from Gödel's Incompleteness Theorems by R.M. Smullyan [1].

2 Athenians and Cretans

2.1 Example for the beginning

Let's imagine a land with 2 groups of people. One group consists of truth-tellers, the other one consists of liars. Further, there exist Athenians, who always tell the truth, and Cretans, who always lie. What would someone have to say to show that he isn't an Athenian, but that he tells the truth? By simply saying: "I'm not an Athenian". A liar could not claim that because he is indeed no Athenian. So his statement has to be true, even if he is not an Athenian.

2.2 Analogy to Gödel's Theorem

This example can easily be converted to one of the main statements of Gödel's Incompleteness Theorem. The truth-teller and liar represent the true and false sentences in a System, Athenians/Cretans represent the provable/refutable sentences. After the statement "I'm not an Athenian", we can conclude that he is a truth-teller, but not an Athenian. Analogous, if there exists a sentence which claims "I'm not provable", this would mean that there are sentences which are true, but not provable. And essentially this is what Gödel's famous theorem is about: "Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F." - Gödel.

3 Diagonalization and Gödel sentences

3.1 Formality

To not get confused by all the letters and function symbols, we use following definitions in our following paper:

- E: set of expressions of L
- S: set of sentences of L
- P: set of provable sentences
- R: set of refutable sentences
- T: set of true sentences
- H: set of predicates

Further, we call a set A expressible in L if A is expressed by some predicate of L:

$$\forall n H(n) \in T \iff n \in A.$$

In words, we say that if the predicate H is satisfied by all of the members of set A, H expresses A.

3.2 Gödel numbers and Diagonalization

To be able to talk about sentences in a formal way, we assign to each expression E a number g(E). This number will be called Gödel number of E. E_n will be called the inverse function of g(E), so it represents the expression with the Gödel number n. E_n(n) will be called the diagonalization of E_n. That sentence is true iff the predicate E_n is satisfied by its own Gödel number. This diagonalization can be visualized as follows:

	0	**1**	**2**	**3**	**4**	**5**	**6**	
E_0	$\boxed{E_0(0)}$	$E_0(1)$	$E_0(2)$	$E_0(3)$	$E_0(4)$	$E_0(5)$	$E_0(6)$	
E_1	$E_1(0)$	$\boxed{E_1(1)}$	$E_1(2)$	$E_1(3)$	$E_1(4)$	$E_1(5)$	$E_1(6)$	
E_2	$E_2(0)$	$E_2(1)$	$\boxed{E_2(2)}$	$E_2(3)$	$E_2(4)$	$E_2(5)$	$E_2(6)$	
E_3	$E_3(0)$	$E_3(1)$	$E_3(2)$	$\boxed{E_3(3)}$	$E_3(4)$	$E_3(5)$	$E_3(6)$	...
E_4	$E_4(0)$	$E_4(1)$	$E_4(2)$	$E_4(3)$	$\boxed{E_4(4)}$	$E_4(5)$	$E_4(6)$	
E_5	$E_5(0)$	$E_5(1)$	$E_5(2)$	$E_5(3)$	$E_5(4)$	$\boxed{E_5(5)}$	$E_5(6)$	
E_6	$E_6(0)$	$E_6(1)$	$E_6(2)$	$E_6(3)$	$E_6(4)$	$E_6(5)$	$\boxed{E_6(6)}$	
				$\vdots$				

The Gödel number of a specific sentence E_n(n) will be called d(n) (diagonal function). For any number set A, by A* we shall mean the set of all numbers n such that the Gödel number of $E_n \in$ A. More formally:

$$n \in A \iff d(n) \in A^*.$$

If we have for example the set A = 1,2,4, then the set A^* would represent the set of the Gödel numbers of $E_1(1)$,$E_2(2)$ and $E_4(4)$. This property will be useful in the next point.

3.3 Final step

The last formal construct to get to our proof is a Gödel sentence. E_n is a Gödel sentence for set A , iff E_n is true and n lies in A.

$$E_\mathrm{n} \in T \iff n \in A$$

If A* is expressible in L, then there is a Gödel sentence for A (Lemma 4.3). With this knowledge we can define our path for proving incompleteness: We need to find a Gödel sentence E_x for $\sim$ P because if there exists one, we see that E_x is true and x lies in ∼P. If we can show that ∼P* is expressible, we know that there exists this Gödel sentence E_x for ∼P, which claims to be not provable itself.

4 Tarski's Theorem for Arithmetic

To further understand the impacts, Gödel's incompleteness will be demonstrated in a system called Peano Arithmetic. The proof for incompleteness is partly based on Tarski's theorem. First, the language and syntax must be defined.

4.1 The Language $\mathcal{L}_E$

The language consists of a set of symbols which are going to be further explained.

$$0 \; ' \; (\;) \; f \quad v \; {}_{\prime} \quad \sim \supset \; \forall \; = \leq \; \sharp$$

0, $0'$, $0''$, $0'''$ are simple numerals for natural numbers 0,1,2,3 and so on.
Every f with a certain amount of dashes represents a function that usually is denoted in a common way. That means '$f_{\prime}$' stands for '+', '$f_{\prime\prime}$' stands for '$\cdot$' and '$f_{\prime\prime\prime}$' stands for 'E' (Exponentiation).
The rest of the symbols are dedicated to a certain function such as '$\sim$' Negation, '$\supset$' Implication, '$\forall$' For All, '$=$' Identity, '$\leq$' Less than or equal.
Variables are denoted with a 'v' and the number of dashes determines every unique one. Examples are ($v_{\prime}$), ($v_{\prime\prime}$), ($v_{\prime\prime\prime}$) and so on.

Considering terms we use following definition:

- Every variable and numeral is a term.
- If t_1 and t_2 are terms, then so are $(t_1 + t_2)$, $(t_1 \cdot t_2), (t_1 E t_2)$ and $t_1{}'$.

A formula has either the form $t_1 = t_2$ or $t_1 \leq t_2$. Furthermore every atomic formula is considered a formula. If F and G are formulas, then $\sim F$ and $F \supset G$ are formulas, and for every variable v_i, the expression $\forall v_i F$ is a formula.

When looking at free and bound occurrences of variables v_i has no free occurrences in $\forall v_i F$. If no variable has any free occurrence in a formula we consider this a sentence.

Substitution of numerals for variables is convenient and is done by writing the numeral and overlining it. The symbol $\overline{n}$ means 0 followed by n primes (Example: $\overline{2}$ is $0''$). When free variables are substituted with any numbers $\overline{k}_1, ..., \overline{k}_n$ then $F[\overline{k}_1, ..., \overline{k}_n]$ is called an instance of $F(v_{i_1}, ..., v_{i_n})$. The square brackets here will be used to express that there has been plugged in a concrete value in the function F, instead of a variable.

Degree is the number of occurrences of logical connectives $\sim$, $\supset$ and the quantifier $\forall$ in a formula.

- Atomic formulas have degree 0.
- Any formulas F and G with degrees d_1 and d_2.
  - $\sim F_1$ has degree $d_1 + 1$.
  - $(F_1 \supset F_2)$ has degree $d_1 + d_2 + 1$.
  - $\forall v_i F_1$ has degree $d_1 + 1$.

With respect to the introduced syntax, we want to use abbreviate forms of certain expressions. In the following part F_1, F_2 are arbitrary formulas t, t_1, t_2 are arbitrary terms and v_i is a variable.

- $(F_1 \vee F_2) = (\sim F_1 \supset F_2)$
- $(F_1 \wedge F_2) = \sim (F_1 \supset \sim F_2)$
- $(F_1 \equiv F_2) = ((F_1 \supset F2) \wedge (F_1 \supset F_2))$
- $(\exists v_i F) = (\sim \forall v_i \sim F)$
- $(t_1 \neq t_2) = (\sim t_1 = t_2)$
- $(t_1 < t_2) = ((t_1 \leq t_2) \wedge (\sim t_1 = t_2))$
- $t_1^{t_2} = t_1 \, E \, t_2$
- $(\forall v_i \leq t)F = \forall v_i(v_i \leq t \supset F)$
- $(\exists v_i \leq t)F = \sim (\forall v_i \leq t) \sim F$

Our desire is to determine what true in our language means. For that matter, the following cases must be applied in order to determine whether or not a formula is true.

- An atomic sentence $c_1 = c_2$ is true if and only if c_1 and c_2 are the same natural number.
- An atomic sentence $c_1 \leq c_2$ is true if and only if the number c_1 is less than or equal to the number c_2.
- A sentence of the form $\sim X$ is true if and only if X is not true.
- A sentence $X \supset Y$ is true if and only if either X is not true or both X and Y are true.
- A sentence $\forall v_i F$ is true if and only if for every number n, the sentence $F[\overline{n}[$ is true.

To clarify what arithmetic means we need to distinguish between two cases one using the small letter '**a**' and one using the capital letter '**A**'.

1. A set or relation is called **A**rithmetic if it is expressed by some formula of $\mathcal{L}_E$.
2. A set or relation is called **a**rithmetic if its expressed by some formula of $\mathcal{L}_E$ in which the exponential symbol 'E' does not occur.

4.2 Concatenation

We define concatenation $x *_b y$ to the Base b for any number $b \geq 2$. To get an idea following example should make it clear how this function works.
The base b for this example is 10, x is 53 and y is 792.

$$53 *_{10} 792 = 53792 = 53 \cdot 10^3 + 792$$

A general form for concatenation looks like this.

$$m *_b n = m \cdot b^{l_b(n)} + n$$

In this case $l_b(n)$ designates the length of n.
We want the concatenation and the results it yields to be a part of our language $\mathcal{L}_E$ leading to:

Proposition 4.1. *For each $b > 2$, the relation $x *_b y = z$ is Arithmetic.*

Proof. Smullyan, Page 21. [1] □

Corollary 4.2. *For each $n \geq 2$, the relation $x_1 *_b x_2 *_b ... *_b x_n = y$ is Arithmetic.*

4.3 Gödel Numbering

We assign a number to every symbol in $\mathcal{L}_E$. That allows us to refer to expressions in $\mathcal{L}_E$ just by referring to their according Gödel Number.

Symbol	0	$'$	(	)	f	$,$	v	$\sim$	$\supset$	$\forall$	$=$	$\leq$	$\sharp$
Number	1	0	2	3	4	5	6	7	8	9	η	ϵ	δ

To further understand how this applies to our syntax let's have a look at an example. The expression

$$(0' f, 0'') \leq (0' f, 0'')$$

needs to be transformed with the according Gödel Numbering. First off let's try to split this expression into it's syntactic atoms. Then we need to write down every corresponding number of each symbol.

(	0	$'$	f	$,$	0	$'$	$'$	)	$\leq$	(	0	$'$	f	$,$	0	$'$	$')$
2	1	0	4	5	1	0	0	3	ϵ	2	1	0	4	5	1	0	0

In Order to find out the Gödel Number now we need to use every number we got as an exponent for a sequence of prime numbers. Consider following formula where ρ is the function that yields a number for a sequence $(a_1, a_2, ..., a_n)$ of numbers. The sequence of numbers $(a_1, a_2, ..., a_n)$ are the Gödel Numbers we get for the symbols that we used in our expression. Let pr_i be the i-th prime number. Then we can define our function ρ:

$$\rho(a_1, a_2, ..., a_n) := \prod_{i=1}^{n} pr_i^{a_i}$$

that gives us the Gödel Number for the input sequence. Now if we apply all our informations we get:

$$2^2 \cdot 3^1 \cdot 5^0 \cdot 7^4 \cdot 11^5 \cdot 13^1 \cdot 17^0 \cdot 19^0 \cdot 23^3 \cdot 29^\epsilon \cdot 31^2 \cdot 37^1 \cdot 41^0 \cdot 43^4 \cdot 47^5 \cdot 53^1 \cdot 61^0 \cdot 67^0$$

which results in

$$13231398892029415770181078049960692151931804840903877$$

and this number can be expressed in our language.

4.4 Tarski's Theorem

Is the set of Gödel numbers of the true sentences in $\mathcal{L}_E$ Arithmetic? Let's recall that a sentence X is a Gödel sentence for a number set A if either X is true and its Gödel number is in A or X is false and its number is not in A. Following Lemma and Theorems can be found in Smullyan, Pages 26-27. [1].

Lemma 4.3. *If A is Arithmetic, then so is A^*.*

Theorem 4.4. *For every Arithmetic set A, there is a Gödel sentence for A.*

Theorem 4.5. ***Tarski's Theorem*** *The set T of Gödel numbers of true Arithmetic sentences is not Arithmetic.*

These theorems will be used in the upcoming part of the paper to show that not all true sentences in a system are provable within the same system. The set of Gödel numbers of the provable sentences is Arithmetic leading to the conclusion that truth and provability don't necessarily come together. By using Theorem 4.4 a true sentence will be shown that is not provable within the system.

5 Example of Incompleteness on Peano arithmetic

5.1 The Peano arithmetic

After the whole theoretical background, we can now come to a concrete example on this. We will be using the Peano Arithmetic, due to its simplicity and intuitiveness. Additionally, we use exponentiation as part of the axioms. This will make the decoding a lot easier, later we will look shortly into the Incompleteness of the Peano Arithmetic without exponentiation. Its axiom system can be divided into 4 groups:

Group I:

L1: $(F \supset (G \supset F))$

L2: $(F \supset (G \supset H)) \supset ((F \supset G) \supset (F \supset H))$

L3: $((\sim F \supset \sim G) \supset (G \supset F))$

Group II:

L4: $(\forall v_i(F \supset G) \supset (\forall v_i F \supset \forall v_i G))$

L5: $(F \supset \forall v_i F)$

L6: $\exists v_i(v_i = t)$

L7: $(v_i = t \supset (X_1 v_i X_2 \supset X_1 t X_2))$

Group III:

N1: $(v_1' = v_2' \supset v_1 = v_2)$

N2: $\sim \overline{0} = v_1'$

N3: $(v_1 + \overline{0}) = v_1$

N4: $(v_1 + v_2') = (v_1 + v_2)'$

N5: $(v_1 \cdot \overline{0}) = \overline{0}$

N6: $(v_1 \cdot v_2') = ((v_1 \cdot v_2) + v_1)$

N7: $(v_1 \leq \overline{0} \equiv v_1 = \overline{0})$

N8: $(v_1 \leq v_2' \equiv (v_1 \leq v_2 \vee v_1 = v_2'))$

N9: $((v_1 \leq v_2) \vee (v_2 \leq v_1))$

N10: $(v_1 \, E \, \overline{0}) = \overline{0}'$

N11: $(v_1 \, E \, v_2') = ((v_1 \, E \, v_2) \cdot v_1)$

Additionally, the Peano arithmetic includes the scheme for mathematical induction, but it will not be used in this paper.
If we talk about proofs in the future, by that we mean a finite sequence of formulas such that each member of the sequence is either an axiom or directly derivable by Modus Ponens (from F and $(F \Rightarrow G)$ we get G) or by Generalization (from F we get $\forall v$ F). If there is a proof whose last member is F, we call F provable. If we can prove the negation of F, we call F refutable.

5.2 Arithmetization

Our goal is to prove that the set of Gödel numbers of the provable formulas of P.E. is an Arithmetic set. For this, we have to construct logical formulas out of our axiom schemes to express this property. To give a short example of how this would work: Let P_E(x) be the property that E_x is provable. We can express P_E(x) as

$$\exists y(Pf(y) \wedge x \in y).$$

$Pf(y)$ here stands for the expression that E_y is a proof. Here we can see an example of how the Gödel numbers are used. Here we can define mathematical properties on these numbers to talk about the properties of a sentence.

Of course, we need to further show that $Pf(y)$ is arithmetic as well, but this will not be covered in this paper (Smullyan, Page 34 [1]).

After the whole theoretical setup and now that we have shown that the set of Gödel numbers of provable formulas of P.E. is Arithmetic we can finally prove the incompleteness of the Peano arithmetic. We let P(v) be a formula, which expresses P_E. Then the formula $\sim$ P(v) expresses the complement of P_E.

Proof. Because we know that if A is expressible, then so is A* (Lemma 4.3). We can find a formula H(v) expressing the set $\sim P_E^*$. Then, its diagonalization $H[\overline{h}]$ is a Gödel sentence for the set $\sim P_E$ (Theorem 4.4). Due to the definition of a Gödel sentence (as a remark here: E_n is true if n lies in $\sim P_E$) we can imply that it is true iff it is not provable in P.E. Because of the correctness of P.E., $H[\overline{h}]$ must be true but not provable in P.E. □

6 Arithmetic Without the Exponential

6.1 The Incompleteness of P.A.

When talking about arithmetic terms and formulas we mean those where the exponential symbol E does not appear. For Peano Arithmetic System (P.A.) we have almost the exact axioms as in P.E. but axioms N_{10} and N_{11} are missing. That's why it's called Peano Arithmetic but without exponentiation. To show incompleteness it's necessary to prove that the relation $x^y = z$ is not only Arithmetic but also arithmetic. For some relations, it's not enough that they are arithmetic but also belong to a set of relations called the Σ_1-relations.

6.2 Σ-formulas

Let n be a natural number and let Σ_n-formulas be first-order formulas in prenex normal form. This formula has a prefix of n alternating blocks of quantifiers that start with an existential block that is followed by a formula which has no quantifiers.

A Σ_1-relation means that the relation is expressible by a Σ_1-formula. This leads us to define what a Σ_1-formula is. But in order to do that we first need to define what Σ_0-formulas are which then will lead to further definitions.
Atomic Σ_0-formulas have one of the following forms: $c_1 + c_2 = c3$, $c_1 \cdot c_2 = c3$, $c_1 = c_2$, $c_1 \leq c2$.

- Every single atomic Σ_0-formula is a Σ_0-formula.
- If F and G are Σ_0-formulas, then $\sim F$ and $F \supset G$ are as well.
- The expression $\forall v_i(v_i \leq c \supset F)$ is a Σ_0-formula where following conditions must hold: F is a Σ_0-formula, v_i is a variable, c is either a numeral or a variable but not v_i.

Σ_1-formulas have following form: $\exists v_{n+1}F(v_1, ..., v_n, v_{n+1})$ where as $F(v_1, ..., v_n, v_{n+1})$ is a Σ_0-formula.

Now with Σ_0-formulas defined we can move on and define Σ-formulas.

- Every Σ_0-formula is a Σ-formula.
- Let F be a Σ-formula. The expression $\exists v_i F$ is a Σ-formula, for any variable v_i.
- Let F be a Σ-formula. The expressions $(\exists v_i \leq v_j)F$ and $(\forall v_i \leq v_j)F$ are Σ-formulas, for v_i and v_j being either variables or numerals.
- Let F and G be Σ-formulas. Then the expressions $F \vee G$ and $F \wedge G$ are Σ-formulas.
- Let F be a Σ_0-formula and G be Σ-formula. Then the expressions $F \supset G$ is a Σ-formula.

6.3 Concatenation to a Prime Base

With P.A. and the exponentiation missing there is a problem to show that the concatenation $x *_p y = z$ is arithmetic. But with the help of an idea due to John Myhill proving that the function is arithmetic it is indeed possible leading to the conclusion that $x *_p y = z$ is arithmetic (Smullyan, Pages 43-45. [1]). Now with concatenation being arithmetic we want to show that $x^y = z$ is firstly arithmetic and in addition is Σ_1.

Theorem 6.1. *The relation $x^y = z$ is Σ_1.*

Proof. Smullyan, Pages 45-47. [1] $\square$

7 The Incompleteness of P.A.

Following corollaries are derived from Theorem 6.1 and can be found in Smullyan, Pages 48-49. [1].

Corollary 7.1. *For any arithmetic set A, the set A^* is arithmetic. If A is Σ, then so is A^*.*

Corollary 7.2. ***Tarski's Theorem** for $\mathcal{L}_A$. The set of Gödel numbers of the true arithmetic sentences is not arithmetic.*

Corollary 7.3. *The sets P_E^* and R_E^* are Σ. The set $\widetilde{P_E^*}$ is arithmetic.*

Now we are going to show the incompleteness of P.A.

Proof. $\widetilde{P_E^*}$ is expressed by an arithmetic formula $H(v_1)$. When diagonalizing this formula we get $H[\overline{h}]$ which is an arithmetic Gödel sentence for $\widetilde{P_E}$ that is true but not provable in P.E. We can conclude that it is true in P.A. but not provable because of the fact that P.A. contains a subset of P.E. axioms. Taking the negation of the sentence $\sim H[\overline{h}]$ which is false that means it is also not provable in P.A. Leading to our final conclusion that $H[\overline{h}]$ is a sentence in $\mathcal{L}_A$ of P.A. which cannot be proven or refuted within P.A. $\square$

8 Conclusion

To understand the impacts of Gödel's incompleteness theorems we would like to give a final example. Take a system for example with a certain amount of axioms. Using the results of the past proofs we can say that we are able to find a true sentence within our system but cannot prove this sentence using the axioms we already have. Because of this, we add this new true sentence as an axiom to expand the system. Now we have a system, which is capable of proving even more sentences, but still, there are infinitely more sentences, which are true and not provable. The only thing we can do is to add these as axioms to get a ”more correct and complete” system, but we will never reach a fully complete system.

We were able to demonstrate incompleteness in one of the most well-known arithmetic systems which is the Peano Arithmetic. This goes to show that incompleteness can also be found in other arithmetics due to the fact that Peano Arithmetic serves as a foundation for them. The result especially underlines the fact that a powerful (regarding the expressibility of arbitrary formulas) axiomatic system isn't necessarily complete within itself. It would be really interesting to see if in the future a different approach to the construction of arithmetic systems may overcome incompleteness.

References

[1] R.M. Smullyan. *Godel's Incompleteness Theorems*. Oxford Logic Guides. Oxford University Press, 1992.