universität innsbruck

Functional Programming	WS 2018/2019	LVA 703024+703025
Exercises 7		December 14, 2018

All Solutions

2 We prove the property by structural induction over xs.

Base Case ($xs = \texttt{[]}$). The base case is shown by the derivation

$$\begin{aligned} \texttt{map}\ f\ (\texttt{map}\ g\ \texttt{[]}) &= \texttt{map}\ f\ \texttt{[]} && \text{by def. of } \texttt{map} \\ &= \texttt{[]} && \text{by def. of } \texttt{map} \\ &= \texttt{map}\ (f \circ g)\ \texttt{[]} && \text{by def. of } \texttt{map} \end{aligned}$$

Step Case ($xs = z : zs$). The IH is

$$\texttt{map}\ f\ (\texttt{map}\ g\ zs) = \texttt{map}\ (f \circ g)\ zs$$

We conclude the step case by the derivation

$$\begin{aligned} \texttt{map}\ f\ (\texttt{map}\ g\ (z : zs)) &= \texttt{map}\ f\ (g\ z : \texttt{map}\ g\ zs) && \text{by def. of } \texttt{map} \\ &= f\ (g\ z) : (\texttt{map}\ f\ (\texttt{map}\ g\ zs)) && \text{by def. of } \texttt{map} \\ &\overset{\text{IH}}{=} f\ (g\ z) : \texttt{map}\ (f \circ g)\ zs && \\ &= \texttt{map}\ (f \circ g)\ (z : zs) && \text{by def. of } \texttt{map} \end{aligned}$$

3 We prove the property by structural induction over xs.

Base Case ($xs = \texttt{[]}$). The base case is shown by the derivation

$$\begin{aligned} \texttt{filter}\ p\ (\texttt{map}\ f\ \texttt{[]}) &= \texttt{filter}\ p\ \texttt{[]} && \text{by def. of } \texttt{map} \\ &= \texttt{[]} && \text{by def. of } \texttt{filter} \\ &= \texttt{map}\ f\ \texttt{[]} && \text{by def. of } \texttt{map} \\ &= \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ \texttt{[]}) && \text{by def. of } \texttt{filter} \end{aligned}$$

Step Case ($xs = z : zs$). The IH is

$$\texttt{filter}\ p\ (\texttt{map}\ f\ zs) = \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ zs)$$

We start the step case by

$$\texttt{filter}\ p\ (\texttt{map}\ f\ (z : zs)) = \texttt{filter}\ p\ (f\ z : \texttt{map}\ f\ zs) \qquad \text{by def. of } \texttt{map}$$

and proceed by a case analysis on whether $p\ (f\ z)$:

- Assume $p\ (f\ z)$ holds. Then we conclude by

$$\begin{aligned} \texttt{filter}\ p\ (f\ z : \texttt{map}\ f\ zs) &= f\ z : \texttt{filter}\ p\ (\texttt{map}\ f\ zs) && \text{by def. of } \texttt{filter} \\ &\overset{\text{IH}}{=} f\ z : \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ zs) && \\ &= \texttt{map}\ f\ (z : \texttt{filter}\ (p \circ f)\ zs) && \text{by def. of } \texttt{map} \\ &= \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ (z : zs)) && \text{by def. of } \texttt{filter} \end{aligned}$$

- Assume $p\ (f\ z)$ does not hold. Then we conclude by

$$\begin{aligned}
\texttt{filter}\ p\ (f\ z : \texttt{map}\ f\ zs) &= \texttt{filter}\ p\ (\texttt{map}\ f\ zs) && \text{by def. of } \texttt{filter} \\
&\overset{\text{IH}}{=} \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ zs) \\
&= \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ (z : zs)) && \text{by def. of } \texttt{filter}
\end{aligned}$$

4 We prove the property by structural induction on xs.

Base Case ($xs = \texttt{[]}$). The base case is shown by the derivation

$$\begin{aligned}
\texttt{map}\ f\ (\texttt{[]} \mathbin{+\!+} ys) &= \texttt{map}\ f\ ys && \text{by def. of } \texttt{map} \\
&= \texttt{[]} \mathbin{+\!+} \texttt{map}\ f\ ys && \text{by def. of } \mathbin{+\!+} \\
&= \texttt{map}\ f\ \texttt{[]} \mathbin{+\!+} \texttt{map}\ f\ ys && \text{by def. of } \texttt{map}
\end{aligned}$$

Step Case ($xs = z : zs$). The IH is

$$\texttt{map}\ f\ (zs \mathbin{+\!+} ys) = \texttt{map}\ f\ zs \mathbin{+\!+} \texttt{map}\ f\ ys$$

We conclude the step case by the derivation

$$\begin{aligned}
\texttt{map}\ f\ ((z : zs) \mathbin{+\!+} ys) &= \texttt{map}\ f\ (z : (zs \mathbin{+\!+} ys)) && \text{by def. of } \mathbin{+\!+} \\
&= f\ z : \texttt{map}\ f\ (zs \mathbin{+\!+} ys) && \text{by def. of } \texttt{map} \\
&\overset{\text{IH}}{=} f\ z : (\texttt{map}\ f\ zs \mathbin{+\!+} \texttt{map}\ f\ ys) \\
&= (f\ z : \texttt{map}\ f\ zs) \mathbin{+\!+} \texttt{map}\ f\ ys && \text{by def. of } \mathbin{+\!+} \\
&= \texttt{map}\ f\ (z : zs) \mathbin{+\!+} \texttt{map}\ f\ ys && \text{by def. of } \texttt{map}
\end{aligned}$$

5 If $n < 0$ the property trivially holds. Thus, we restrict n to be a natural number and prove the property for arbitrary xs by induction on n.

Base Case ($n = 0$). The base case is shown by the derivation

$$\begin{aligned}
\texttt{take}\ 0\ (\texttt{map}\ f\ xs) &= \texttt{[]} && \text{by def. of } \texttt{take} \\
&= \texttt{map}\ f\ \texttt{[]} && \text{by def. of } \texttt{map} \\
&= \texttt{map}\ f\ (\texttt{take}\ 0\ xs) && \text{by def. of } \texttt{take}
\end{aligned}$$

Step Case ($n = k + 1$). The IH is

$$\forall xs.\, \texttt{take}\ k\ (\texttt{map}\ f\ xs) = \texttt{map}\ f\ (\texttt{take}\ k\ xs)$$

Now if $xs = \texttt{[]}$ the property trivially holds, thus let us assume $xs = z : zs$.

$$\begin{aligned}
\texttt{take}\ (k+1)\ (\texttt{map}\ f\ (z : zs)) &= \texttt{take}\ (k+1)\ (f\ z : \texttt{map}\ f\ zs) && \text{by def. of } \texttt{map} \\
&= f\ z : \texttt{take}\ k\ (\texttt{map}\ f\ zs) && \text{by def. of } \texttt{take} \\
&\overset{\text{IH}}{=} f\ z : \texttt{map}\ f\ (\texttt{take}\ k\ zs) \\
&= \texttt{map}\ f\ (z : \texttt{take}\ k\ zs) && \text{by def. of } \texttt{map} \\
&= \texttt{map}\ f\ (\texttt{take}\ (k+1)\ (z : zs)) && \text{by def. of } \texttt{take}
\end{aligned}$$

6 If $n < 0$ the property trivially holds. Thus, we restrict n to be a natural number and and prove the property for arbitrary xs by induction on n.

Base Case $(n = 0)$. The base case is shown by the derivation

$$\begin{aligned} \texttt{take}\ 0\ xs \mathbin{+\!+} \texttt{drop}\ 0\ xs &= \texttt{[]} \mathbin{+\!+} xs && \text{by def. of } \texttt{take} \text{ and } \texttt{drop} \\ &= xs && \text{by def. of } \mathbin{+\!+} \end{aligned}$$

Step Case $(n = k + 1)$. The IH is

$$\forall xs.\, \texttt{take}\ k\ xs \mathbin{+\!+} \texttt{drop}\ k\ xs = xs$$

Now if $xs = \texttt{[]}$ the property trivially holds, thus let us assume $xs = z : zs$.

$$\begin{aligned} \texttt{take}\ (k+1)\ (z : zs) \mathbin{+\!+} \texttt{drop}\ (k+1)\ (z : zs) &= (z : \texttt{take}\ k\ zs) \mathbin{+\!+} \texttt{drop}\ (k+1)\ (z : zs) && \text{by def. of } \texttt{take} \\ &= z : (\texttt{take}\ k\ zs \mathbin{+\!+} \texttt{drop}\ (k+1)\ (z : zs)) && \text{by def. of } \mathbin{+\!+} \\ &= z : (\texttt{take}\ k\ zs \mathbin{+\!+} \texttt{drop}\ k\ zs) && \text{by def. of } \texttt{drop} \\ &\overset{\text{IH}}{=} z : zs \end{aligned}$$