



Functional Programming

Lecture 5

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Topics

abstract data types, algebraic data types, binary search trees, combinator parsing, efficiency, encoding data types as lambda-terms, evaluation strategies, formal verification, first steps, guarded recursion, Haskell introduction, higher-order functions, historical overview, induction, infinite data structures, input and output, lambda-calculus, lazy evaluation, list comprehensions, lists, modules, pattern matching, polymorphism, property-based testing, reasoning about functional programs, recursive functions, sets, strings, tail recursion, trees, tupling, type checking, type inference, types, types and type classes, unification, user-defined types

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Overview

- Introduction to the λ -Calculus
- Encoding Data Types

Introduction to the λ -Calculus

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- search for general framework in which every algorithm can be defined

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- that is, **Turing-complete** is same as **definable in λ -calculus**

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- and shows both models of computation equivalent
- that is, Turing-complete is same as definable in λ -calculus
- λ -calculus closer to functional programming

Syntax – λ -Terms

- grammar

$t ::= x$	variable/atom
$(t t)$	application
$(\lambda x. t)$	(lambda) abstraction

- occurrences of x in $\lambda x. t$ are **bound** (by **binder** “ $\lambda x.$ ”)

Syntax – λ -Terms

- grammar

$$\begin{array}{ll} t ::= x & \text{variable/atom} \\ | (t t) & \text{application} \\ | (\lambda x. t) & \text{(lambda) abstraction} \end{array}$$

- occurrences of x in $\lambda x. t$ are bound (by binder “ $\lambda x.$ ”)

Examples

$$(\lambda x. y)$$
$$(\lambda x. (\lambda y. x))$$
$$(\lambda x. (\lambda y. (\lambda z. ((x z) (y z)))))$$
$$(\lambda x. ((\lambda y. (\lambda z. (z y))) x))$$

Conventions (to ease reading and writing)

Examples (continued)

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Conventions (to ease reading and writing)

- outermost parentheses are dropped
- application binds stronger than abstraction, e.g., $\lambda x. x z$ is equal to $\lambda x. (x z)$ and not to $(\lambda x. x) z$

Examples (continued)

$$\lambda x. y$$
$$\lambda x. (\lambda y. x)$$
$$\lambda x. (\lambda y. (\lambda z. (x z) (y z)))$$
$$\lambda x. (\lambda y. (\lambda z. z y)) x$$

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- application associates to the left

Examples (continued)

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- application associates to the left
- nested abstractions associate to the right and variables are combined

Examples (using Conventions)

$\lambda x. y$

$\lambda xy. x$

$\lambda xyz. x z (y z)$

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Note

- think of nested lambdas as “functions with multiple arguments”
- e.g., $\lambda xyz. t$ is function taking 3 arguments

Haskell vs. λ -Terms

Haskell

- `\x -> x + 1`

λ -Calculus

- $\lambda x. \text{add } x \ 1$

Haskell vs. λ -Terms

Haskell

- $\backslash x \rightarrow x + 1$
- $(\backslash x \rightarrow x + 1) 2$ $(\rightsquigarrow 3)$

λ -Calculus

- $\lambda x. \text{add } x 1$
- $(\lambda x. \text{add } x 1) 2$

Haskell vs. λ -Terms

Haskell

- `\x -> x + 1`
- `(\x -> x + 1) 2` (\rightsquigarrow 3)
- `if True then 1 else 0` (\rightsquigarrow 1)

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- $(\lambda x. \text{add } x \ 1) \ 2$
- `ite True 1 0`

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- `\x -> x + 1`
- `(\x -> x + 1) 2` (\rightsquigarrow 3)
- `if True then 1 else 0` (\rightsquigarrow 1)
- `(,) 2 4` (\rightsquigarrow (2, 4))

λ -Calculus

- $\lambda x. \text{add } x \ 1$
- $(\lambda x. \text{add } x \ 1) \ 2$
- `ite True 1 0`
- `Pair 2 4`

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- `(\x -> x + 1) 2` (\rightsquigarrow 3)
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- `(,) 2 4` (\rightsquigarrow (2, 4))
- `fst (2, 4)` (\rightsquigarrow 2)

λ -Calculus

- $\lambda x. \text{add } x \ 1$
- $(\lambda x. \text{add } x \ 1) \ 2$
- `ite True 1 0`
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- `\x -> x + 1`
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λ -Calculus

- $\lambda x. \text{add } x \ 1$
- $(\lambda x. \text{add } x \ 1) \ 2$
- `ite True 1 0`
- `Pair 2 4`
- `fst (Pair 2 4)`

Remark

- '0', '1', '2', '4', 'add', 'fst', 'ite', 'Pair', and 'True' abbreviate more complex λ -terms
- supposed to “encode” behavior of 0, 1, 2, 4, (+), `fst`, `if ... then ... else`, `(,)`, and `True`

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- given $f(x) = x + 5$
- compute $f(3)$ by substituting 3 for x in $x + 5$, written $(x + 5)[x := 3]$

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Single-Step β -Reduction (semi-formal definition)

- intuition: apply “function” to “argument”
- in λ -calculus both “functions” and “arguments” are λ -terms
- the β -rule

$$(\lambda x. s) t \rightarrow_{\beta} s[x := t]$$

- in words: *when applying function $(\lambda x. s)$ to input t , replace every occurrence of x in body of function (that is, s) by t*

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substitution replacing occurrences of x by t

- in words: *when applying function $(\lambda x. s)$ to input t , replace every occurrence of x in body of function (that is, s) by t*

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$$(\lambda x. s) t \rightarrow_{\beta} s[x := t]$$

Examples

$$(\lambda x. x) (\lambda x. x)$$

$$(\lambda x y. y) (\lambda x. x)$$

$$(\lambda x y z. x z (y z)) (\lambda x. x)$$

$$(\lambda x. x x) (\lambda x. x x)$$

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$\lambda x. x$

no β -step possible

Binders are Everywhere

- logic: for all $\forall x. \phi(x)$, exists $\exists x. \psi(x)$, ...
- calculus: derivative $\frac{d}{dx} f$, integral $\int f dx$, ...
- combinatorics: sum $\sum_{i=1}^n g(i)$, product $\prod_{j=1}^n h(j)$, ...

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- clearly not intended (problem was that **free** y was **bound/captured** when substituting for x)

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Solution

- do not allow arbitrary substitution (**freshness constraints**)

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Solution

- do not allow arbitrary substitution (freshness constraints)
- work modulo **consistent renaming** of bound variables (α -equivalence)

Renaming Variables with Permutations

- **permutation** is bijection between variables (can be represented by finite composition of swappings $(x_n \rightleftharpoons y_n) \circ \cdots \circ (x_1 \rightleftharpoons y_1)$)

Renaming Variables with Permutations

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- **apply permutation** π to term t , written $\pi \cdot t$

$$\pi \cdot x = \pi(x)$$

$$\pi \cdot (t u) = (\pi \cdot t) (\pi \cdot u)$$

$$\pi \cdot (\lambda x. t) = \lambda \pi(x). (\pi \cdot t)$$

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Example

- $(x \rightleftharpoons y) \cdot (\lambda xy. x (\lambda zx. z x))$

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$$\pi \cdot x = \pi(x)$$

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Free Variables and Freshness

- set of **free variables** of a term

$$\mathcal{F}(x) = \{x\}$$

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- α -equivalence class of $\lambda x. y x$ (remember: all elements considered the same term)

$$\{\lambda x. y x, \quad \lambda z. y z, \quad \lambda a. y a, \quad \lambda x'. y x', \quad \lambda b_1. y b_1, \quad \dots\}$$

Substitutions

- substitution (for terms) is function from variables to terms
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Example

- consider $\sigma = [x := \lambda x. x]$
- then $\sigma(x) = \lambda x. x$ and
- $\sigma(y) = y$ for all $y \neq x$

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Single-Step β -Reduction (formal definition)

- inductive definition of one step β -reduction

$$\frac{}{(\lambda x. t) u \rightarrow_{\beta} t[x := u]} \text{ (root)} \qquad \frac{s \rightarrow_{\beta} t}{s u \rightarrow_{\beta} t u} \text{ (app-l)}$$
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- inductive definition of one step β -reduction

$$\frac{}{(\lambda x. t) u \rightarrow_{\beta} t[x := u]} \text{ (root)} \qquad \frac{s \rightarrow_{\beta} t}{s u \rightarrow_{\beta} t u} \text{ (app-l)}$$

$$\frac{s \rightarrow_{\beta} t}{\lambda x. s \rightarrow_{\beta} \lambda x. t} \text{ (abs)} \qquad \frac{s \rightarrow_{\beta} t}{u s \rightarrow_{\beta} u t} \text{ (app-r)}$$

- intuition: if s has subterm of form $(\lambda x. t) u$, then replacing it by $t[x := u]$ is a β -step
- we call $(\lambda x. t) u$ a redex (short for reducible expression), and
- $t[x := u]$ its contractum
- $s \rightarrow_{\beta}^* t$ (with \rightarrow_{β}^* reflexive, transitive closure of \rightarrow_{β}) denotes existence of sequence $s = t_1 \rightarrow_{\beta} t_2 \rightarrow_{\beta} \cdots \rightarrow_{\beta} t_n = t$ with $n \geq 0$
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(say “ s (β -)reduces to t ”)
- existence of nonempty sequence (that is, $n > 0$) denoted by $s \rightarrow_{\beta}^+ t$

Exercise

- consider $\Omega = (\lambda x. x x) (\lambda x. x x)$,
- $K = \lambda xy. x$,
- $\lambda xy. y$, and
- $I = \lambda x. x$
- reduce the following λ -terms

$$K \Omega$$

$$(\lambda xy. y) \Omega$$

$$I \Omega$$

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Examples

- $\lambda x. x$ is in NF
- $(\lambda x. x) y$ is not in NF, since $(\lambda x. x) y \rightarrow_{\beta} y$ (with NF y)

Encoding Data Types

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Haskell

- `True`
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λ -Calculus

- $\text{True} = \lambda xy. x$ “ignore second argument”
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ite True x y
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Natural Numbers – Church Numerals

- define n -fold function application

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function that applies first argument n -times to second argument

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Haskell vs. λ -Calculus

Haskell

- 0
- 1
- n
- (+)
- (*)
- (^)

λ -Calculus

- $0 = \lambda f x. x$
- $1 = \lambda f x. f x$
- $n = \lambda f x. f^n x$
- $\text{add} = \lambda m n f x. m f (n f x)$
- $\text{mul} = \lambda m n f. m (n f)$
- $\text{exp} = \lambda m n. n m$

Pairs

Haskell

- `(,)`
- `fst`
- `snd`

λ -Calculus

- $\text{Pair} = \lambda xyf. f\ x\ y$
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Lists

Haskell

- `(:)`
- `head`
- `tail`
- `[]`
- `null`

λ -Calculus

- $\text{Cons} = \lambda xy. \text{Pair False} (\text{Pair } x\ y)$
- $\text{head} = \lambda z. \text{fst} (\text{snd } z)$
- $\text{tail} = \lambda z. \text{snd} (\text{snd } z)$
- $\text{Nil} = \lambda x. x$
- $\text{null} = \text{fst}$

Recursion

- Haskell function

```
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           else 1 + length (tail x)
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Example – Length

- recall that $\text{length} = Y g$ with $g = \lambda f x. \text{ite} (\text{null } x) 0 (\text{add } 1 (f (\text{tail } x)))$
- by fixed point property we obtain $\text{length} \rightarrow_{\beta}^* g \text{ length}$, taking care of replacing additional parameter f in g by definition of length

Exercise Preparation – Instantiating Type Classes

general schema for turning type T into instance of type class C

instance C T **where**

```
-- implementations of class functions
```


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general schema for turning type `T` into instance of type class `C`

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instance C T where
```

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```

Example – Equality for User-Defined Type

- consider type `data YNM = Yes | No | Maybe`
- instance declaration

```
instance Eq YNM where
```

```
Yes    == Yes    = True
```

```
No     == No     = True
```

```
Maybe == Maybe  = True
```

```
  _    ==    _    = False
```

Homework (for November 30th)

1. Read the [lecture notes on the lambda calculus](#) until Section 5.
2. Encode the Haskell function `boolToInt :: Bool -> Int`

```
boolToInt True = 1
```

```
boolToInt False = 0
```

as λ -term and step-wise compute the NF of the λ -term encoding `boolToInt True`.

3. Give a combinator E (“eraser”) that satisfies $E t \rightarrow_{\beta}^* E$ for arbitrary λ -terms t .

Hint: Fix $E = A A$ and try to come up with an appropriate A .

4. Given the following Haskell type for λ -terms

```
type Id = String
```

```
data Term = Var Id | App Term Term | Abs Id Term
```

implement a function `redexes :: Term -> [Term]` that computes the list of all redexes occurring in a term.

Homework (for November 30th, continued)

5. Write a `Show` class instance for the `Term` type of Exercise 4 that produces pretty output conforming to the conventions of slide 7.

Example:

```
show (App (App (Var "x") (Abs "y" (Var "y")))) (Var "z"))
= "x (\y. y) z"
```

Hint: Consult the official documentation of `Show`, `showsPrec`, `showString`, and `showParen`.

6. Implement a function

```
subst :: String -> Term -> Term -> Term
```

such that `subst x s t` computes `t[x := s]` as given on slide 16.
(Bonus: satisfy arising freshness constraints on the fly.)