



Functional Programming

Lecture 6

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Topics

abstract data types, algebraic data types, binary search trees, combinator parsing, efficiency, encoding data types as lambda-terms, evaluation strategies, formal verification, first steps, guarded recursion, Haskell introduction, higher-order functions, historical overview, induction, infinite data structures, input and output, lambda-calculus, lazy evaluation, list comprehensions, lists, modules, pattern matching, polymorphism, property-based testing, reasoning about functional programs, recursive functions, sets, strings, tail recursion, trees, tupling, type checking, type inference, types, types and type classes, unification, user-defined types

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Overview

- Evaluation Strategies
- Abstract Data Types
- Sets as Binary Search Trees

Evaluation Strategies

Recall λ -Terms

$$t ::= x \mid (t t) \mid (\lambda x. t)$$

Recall λ -Terms

$$t ::= x \mid (t t) \mid (\lambda x. t)$$

Examples

conventions	verbose	in words
$x y$	$(x y)$	" x applied to y "
$\lambda x. x$	$(\lambda x. x)$	"lambda x to x " (identity function)
$\lambda x y. x$	$(\lambda x. (\lambda y. x))$	"lambda $x y$ to x "
$\lambda x. x x$	$(\lambda x. (x x))$	"lambda x to x applied to x "
$(\lambda x. x) x$	$((\lambda x. x) x)$	"lambda x to x , applied to x "

Recall β -Reduction

- term s (β -)reduces to term t in one step
- written: $s \rightarrow_{\beta} t$
- if there is **redex** $(\lambda x. u) v$ in s such that
- replacing $(\lambda x. u) v$ in s by **contractum** $u[x := v]$ results in t

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- with exponentiation $\text{exp} = \lambda m n. n m$
- compute “one to the power of one”
 $\text{exp } 1 \ 1$

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- with exponentiation $\text{exp} = \lambda m n. n m$
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$$\begin{aligned} \text{exp } 1 \ 1 &= (\lambda m n. n m) \ 1 \ 1 \\ &\rightarrow_{\beta} (\lambda n. n \ 1) \ 1 \end{aligned}$$

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$$\rightarrow_{\beta} \lambda x. \lambda x. x \ x \quad \times \text{ variable capture!}$$

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Evaluation Order

- consider $d\ x = x + x$
- term $d\ (d\ 2)$ may be evaluated as follows

$d\ (d\ 2)$

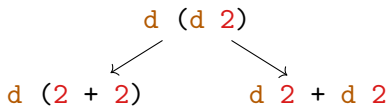
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$$\begin{array}{c} d\ (d\ 2) \\ \swarrow \\ d\ (2 + 2) \end{array}$$

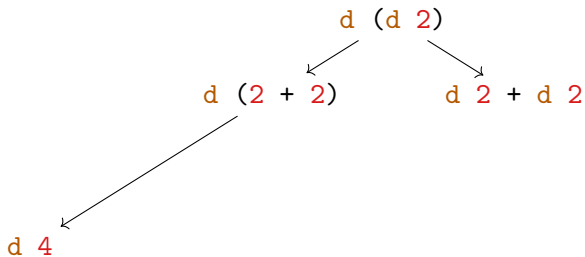
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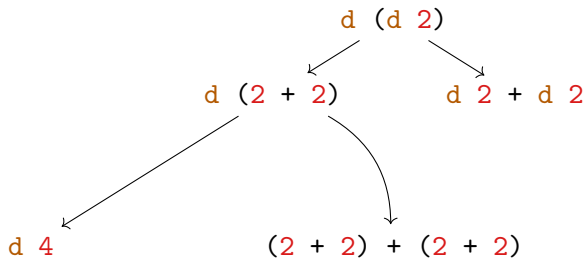
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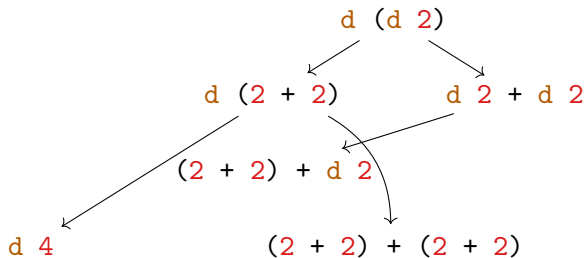
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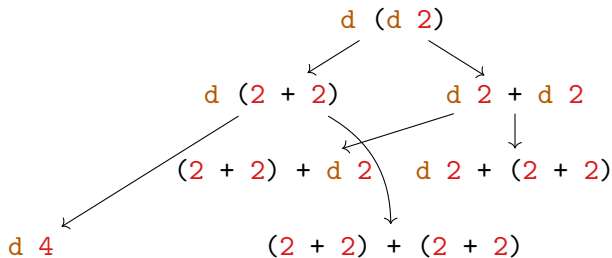
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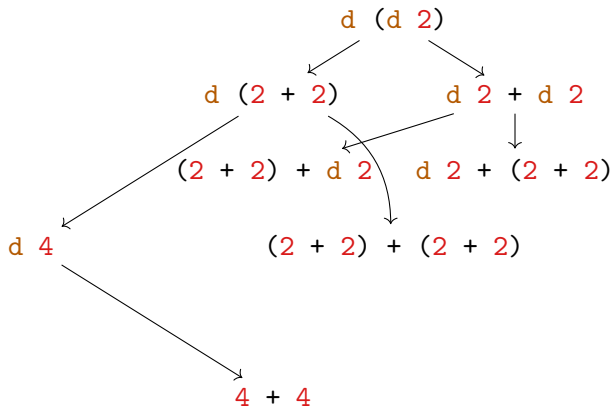
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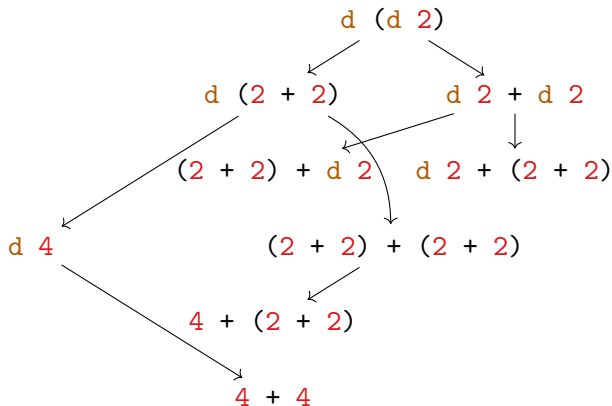
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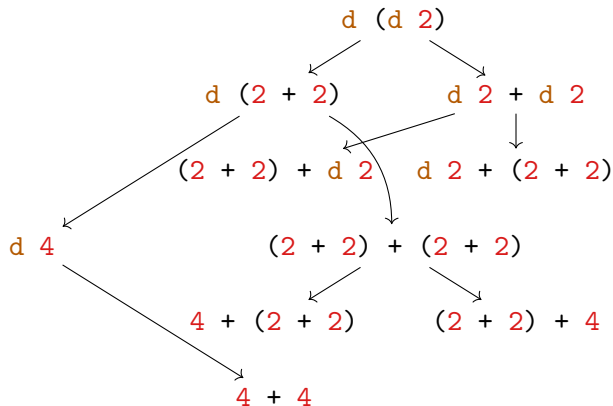
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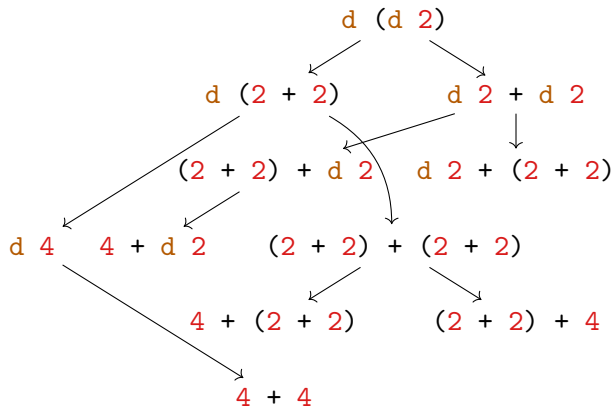
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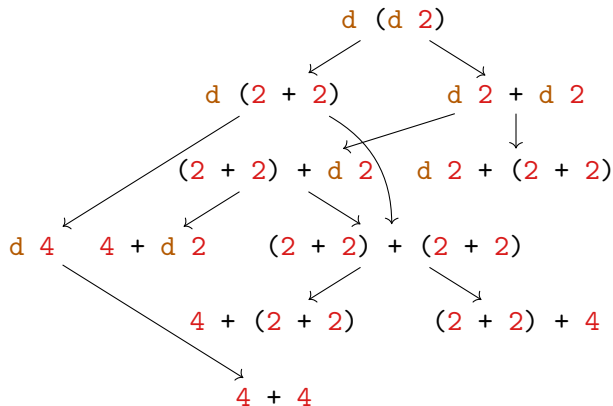
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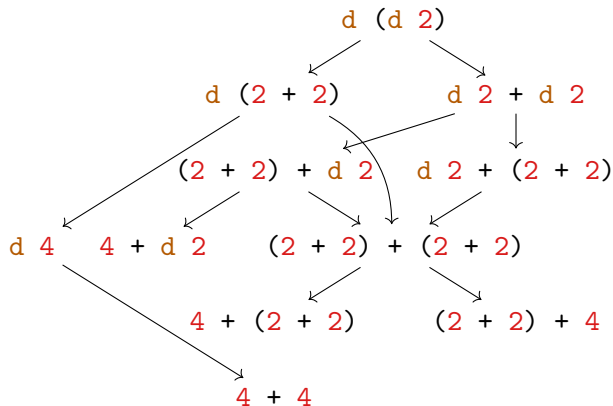
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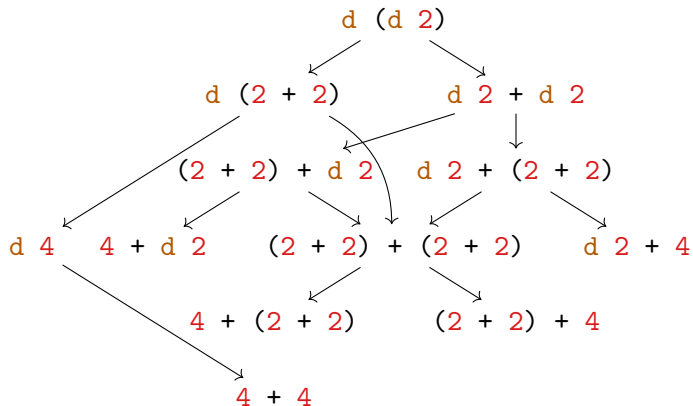
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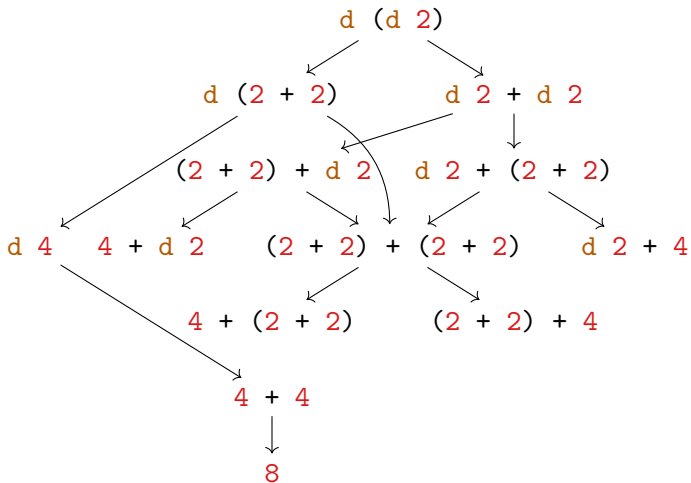
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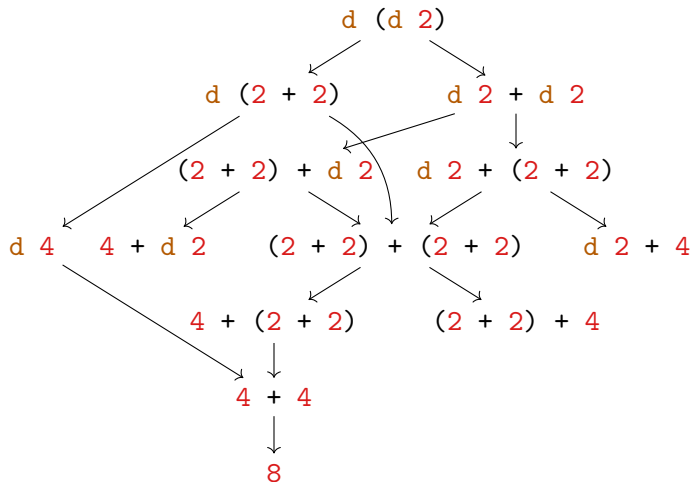
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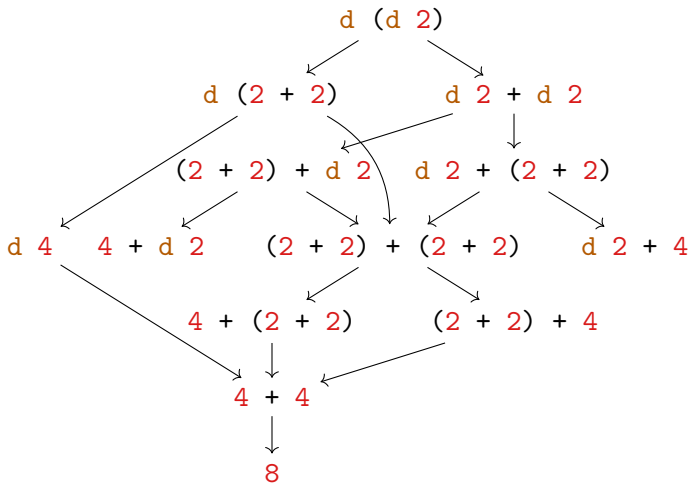
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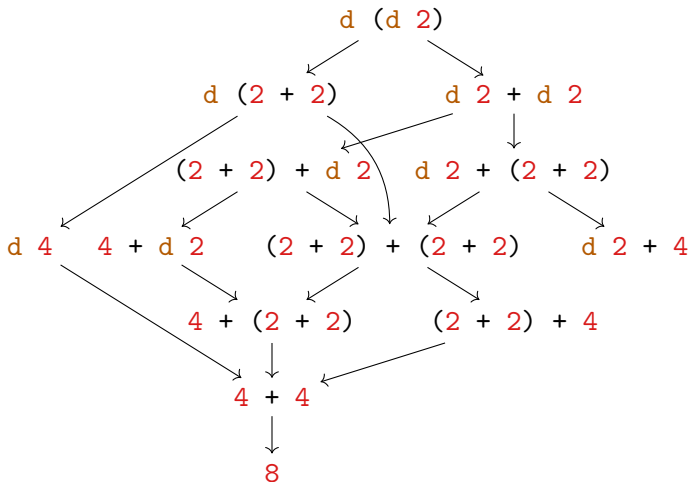
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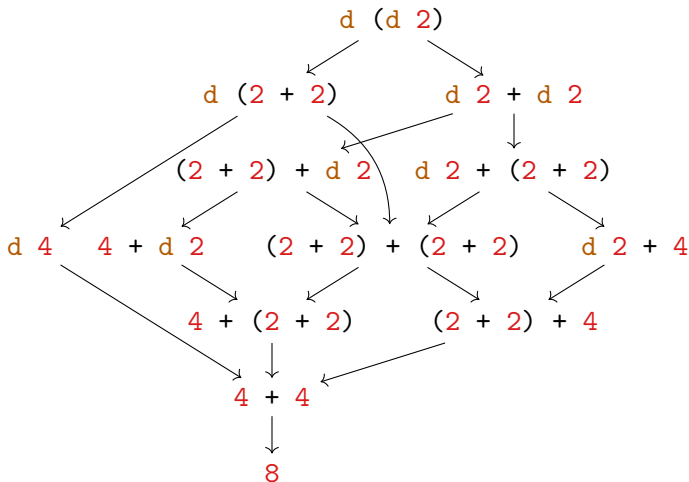
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(Reduction) Strategies

- fix evaluation order
- call by value (idea: compute arguments before function calls)
- call by name (idea: pass expressions rather than their results)

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what is called **evaluation strategy** in programming, is typically called **reduction strategy** in λ -calculus

(Reduction) Strategies

- fix evaluation order
- call by value (idea: compute arguments before function calls)
- call by name (idea: pass expressions rather than their results)

Example

- call by value

d (d 2)

- call by name

d (d 2)

(Reduction) Strategies

- fix evaluation order
- call by value (idea: compute arguments before function calls)
- call by name (idea: pass expressions rather than their results)

Example

- call by value

$d (d 2) = d (2 + 2)$

- call by name

$d (d 2)$

(Reduction) Strategies

- fix evaluation order
- call by value (idea: compute arguments before function calls)
- call by name (idea: pass expressions rather than their results)

Example

- call by value

$$\begin{aligned}d (d 2) &= d (2 + 2) \\ &= d 4\end{aligned}$$

- call by name

$$d (d 2)$$

(Reduction) Strategies

- fix evaluation order
- call by value (idea: compute arguments before function calls)
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Example

- call by value

$$\begin{aligned}d \ (d \ 2) &= d \ (2 + 2) \\ &= d \ 4 \\ &= 4 + 4\end{aligned}$$

- call by name

$$d \ (d \ 2)$$

(Reduction) Strategies

- fix evaluation order
- call by value (idea: compute arguments before function calls)
- call by name (idea: pass expressions rather than their results)

Example

- call by value

$$\begin{aligned}d (d 2) &= d (2 + 2) \\ &= d 4 \\ &= 4 + 4 \\ &= 8\end{aligned}$$

- call by name

$$d (d 2)$$

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$$d (d 2) = d 2 + d 2$$

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$$\begin{aligned}d(d\ 2) &= d\ 2 + d\ 2 \\ &= (2 + 2) + d\ 2\end{aligned}$$

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- call by name

$$\begin{aligned}d \ (d \ 2) &= d \ 2 + d \ 2 \\ &= (2 + 2) + d \ 2 \\ &= 4 + d \ 2 \\ &= 4 + (2 + 2)\end{aligned}$$

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- call by value (idea: compute arguments before function calls)
- call by name (idea: pass expressions rather than their results)

Example

- call by value

$$\begin{aligned}d \ (d \ 2) &= d \ (2 + 2) \\ &= d \ 4 \\ &= 4 + 4 \\ &= 8\end{aligned}$$

- call by name

$$\begin{aligned}d \ (d \ 2) &= d \ 2 + d \ 2 \\ &= (2 + 2) + d \ 2 \\ &= 4 + d \ 2 \\ &= 4 + (2 + 2) \\ &= 4 + 4 \\ &= 8\end{aligned}$$

Applicative Order Reduction

- reduce rightmost innermost redex
- redex is **innermost** if it does not contain redexes itself

Applicative Order Reduction

- reduce rightmost innermost redex
- redex is innermost if it does not contain redexes itself

Example

- consider $t = (\lambda x. (\lambda y. y) x) z$
- $(\lambda y. y) x$ is innermost redex
- t is redex, but not innermost

Normal Order Reduction

- reduce leftmost outermost redex
- redex is **outermost** if it is not contained in another redex

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- redex is outermost if it is not contained in another redex

Example

- consider $t = (\lambda x. (\lambda y. y) x) z$
- t is outermost redex
- $(\lambda y. y) x$ is redex, but not outermost

Exercises

- consider the λ -terms
- $S = \lambda xyz. x z (y z)$
- $K = \lambda xy. x$
- $I = \lambda x. x$
- reduce $S K I$ to NF using applicative order reduction
- reduce $S K I$ to NF using normal order reduction

Further Classification of λ -Terms

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 - $\text{whnf}(x) = \text{true}$
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Examples

term t	value	WHNF
$(\lambda x. x) x$		
$x y$		
x		
$\lambda x. (\lambda y. y) x$		

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Examples

term t	value	WHNF
$(\lambda x. x) x$	X	
$x y$		
x		
$\lambda x. (\lambda y. y) x$		

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Examples

term t	value	WHNF
$(\lambda x. x) x$	✗	✗
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- stop at values
- otherwise choose outermost redex whose right-hand side is value
- corresponds to strict (or eager) evaluation
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Call by Name

- stop at WHNFs
- otherwise same as normal order (that is, leftmost outermost redex)
- corresponds to lazy evaluation (without memoization)
- adopted for example by Haskell

Abstract Data Types

Idea

- hide implementation details
- just provide interface
- allows us to change implementation (e.g., make more efficient) without breaking client code

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Haskell

- consider module
`module M (T, ...) where`
`data T = C1 | ... | CN`
- only name `T` is exported, but none of constructors `C1` to `CN`
- thus we are not able to directly construct values of type `T`
- if we want to export `C1` to `CN`, we can use `T(..)` in export list

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Examples

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Operations on Sets

description	notation	Haskell
empty set	\emptyset	<code>empty :: Set a</code>
insertion	$\{x\} \cup S$	<code>insert :: a -> Set a -> Set a</code>
membership	$e \in S$	<code>mem :: a -> Set a -> Bool</code>
union	$S \cup T$	<code>union :: Set a -> Set a -> Set a</code>
difference	$S \setminus T$	<code>diff :: Set a -> Set a -> Set a</code>

Example – Sets as Lists

```
module Set (Set,empty,insert,mem,union,diff,...) where
import qualified Data.List as List
data Set a = Set [a]
```

```
empty :: Set a
empty = Set []
```

```
insert :: Eq a => a -> Set a -> Set a
insert x (Set xs) = Set $ List.nub $ x : xs
```

```
mem :: Eq a => a -> Set a -> Bool
x `mem` Set xs = x `elem` xs
```

```
union, diff :: Eq a => Set a -> Set a -> Set a
union (Set xs) (Set ys) = Set $ List.nub $ xs ++ ys
diff (Set xs) (Set ys) = Set $ xs List.\\ ys
```

New Types

- `data` with single constructor `Set` used to hide implementation
- in this common special case use `newtype` `Set a = Set [a]` instead
- only difference: `newtype` has better performance than `data`

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Record Syntax

- for data type / new type `T`, instead of `C t1 ... tN`, we may use
- `C {n1 :: t1, ..., nN :: tN}` as constructor
- provides selector functions `n1 :: T -> t1, ..., nN :: T -> tN`

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Example

- **data** `Equation a = E { lhs :: a, rhs :: a }`

```
ghci> let e1 = E "10" "5+5"
```

```
ghci> let e2 = E { rhs = "5+5", lhs = "10" }
```

```
ghci> lhs e1
```

```
"10"
```

```
ghci> rhs e2
```

```
"5+5"
```


Sets as Binary Search Trees

The Type

- use type `BTree` without prefix: `import BTree (BTree(...))`
- import remaining functions from `BTree` with prefix
`import qualified BTree`
- internal representation of set is binary tree (with selector `rep`)
`newtype Set a = Set { rep :: BTree a }`

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`newtype Set a = Set { rep :: BTree a }`

Note

- `newtype Set a = Set { rep :: BTree a }` is almost same as writing `type Set a = BTree a`
- additionally type system prevents “accidental” (that is, without constructor `Set`) use of `BTrees` as `Sets`
- no runtime penalty (in contrast to
`data Set a = Set { rep :: BTree }`)
- reason: `newtype` restricted to **single** constructor (usually of same name as newly introduced type)
- `data` may have arbitrarily many constructors (e.g., `Empty` and `Node`)

Empty Set

```
empty :: Set a  
empty = Set Empty
```

Empty Set

```
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```

Membership

```
mem :: Ord a => a -> Set a -> Bool
x `mem` s = x `memTree` rep s
  where
    memTree x Empty = False
    memTree x (Node y l r) =
      case compare x y of
        EQ -> True
        LT -> x `memTree` l
        GT -> x `memTree` r
```

Insertion

```
insert :: Ord a => a -> Set a -> Set a
insert x s = Set $ insertTree x $ rep s
```

```
insertTree :: Ord a => a -> BTree a -> BTree a
insertTree x Empty          = Node x Empty Empty
insertTree x (Node y l r) =
  case compare x y of
    EQ -> Node y l r
    LT -> Node y (insertTree x l) r
    GT -> Node y l (insertTree x r)
```

Union

```
union :: Ord a => Set a -> Set a -> Set a
union s t = Set $ rep s `unionTree` rep t
```

```
unionTree :: Ord a => BTree a -> BTree a -> BTree a
unionTree Empty s          = s
unionTree (Node x l r) s =
  insertTree x $ l `unionTree` r `unionTree` s
```

Removing the Maximal Element

```
splitMaxFromTree :: BTree a -> Maybe (a, BTree a)
splitMaxFromTree Empty                = Nothing
splitMaxFromTree (Node x l Empty)     = Just (x, l)
splitMaxFromTree (Node x l r)        =
  let Just (m, r') = splitMaxFromTree r
  in Just (m, Node x l r')
```


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The Maybe Type

- Prelude: `data Maybe a = Just a | Nothing`
- used for type-based error handling
- if an error occurs, we return `Nothing`
- otherwise `Just` the result

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Example – Safe Head

```
safeHead (x:_) = Just x
safeHead _     = Nothing
```

Remove Given Element

```
removeFromTree :: Ord a => a -> BTree a -> BTree a
removeFromTree x Empty          = Empty
removeFromTree x (Node y l r) = case compare x y of
  LT -> Node y (removeFromTree x l) r
  GT -> Node y l (removeFromTree x r)
  EQ -> case splitMaxFromTree l of
    Nothing      -> r
    Just (m, l') -> Node m l' r
```

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  GT -> Node y l (removeFromTree x r)
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    Nothing    -> r
    Just (m, l') -> Node m l' r
```

For Binary Search Tree (BST)

- x smaller y : x can only occur in l
- x greater y : x can only occur in r
- x equals y : remove current node and
- combine l and r into new BST
- therefore, take maximum of l as new root
- guarantees that all other elements in l are smaller and
- that all elements in r are greater

Difference

```
diff :: Ord a => Set a -> Set a -> Set a
diff s t = Set $ rep s `diffTree` rep t
```

```
diffTree :: Ord a => BTree a -> BTree a -> BTree a
diffTree t Empty          = t
diffTree t (Node x l r) =
  removeFromTree x t `diffTree` l `diffTree` r
```

Homework (for December 7th)

1. Read Section 5 of the [lecture notes on the lambda calculus](#).
2. Implement safe versions (that is, avoiding runtime errors using type `Maybe`) of `tail`, `init`, and `last`.

Examples:

- `safeTail [] = Nothing`
- `safeLast [1..10] = Just 10`

3. Reduce each of the following λ -terms to NF, once using applicative order reduction and once using normal order reduction:

$$(\lambda w. w) ((\lambda x y. y) (z z))$$
$$(\lambda x y. x) (\lambda z. y z)$$
$$\lambda z. (\lambda x. x z y) (\lambda x y. y z)$$
$$\lambda x y. y (\lambda w. w) (\lambda y z. y x)$$

Homework (for December 7th, continued)

4. Implement single-step call by name reduction as a function `cbn :: Term -> Maybe Term` (where the result is `Nothing` if no step is possible).
Hint: Start by implementing `isWHNF :: Term -> Bool` (checking whether a term is in WHNF) and `root :: Term -> Maybe Term` (performing a single root β -step, if possible).
5. Write an `Eq` class instance for the `BTree` based `Set` type of this lecture (see also `Set.hs`).
6. Give a `Set` implementation (supporting all functions of slide 16 based on sorted lists. Make sure that each operation preserves the invariant that the internal representation is in fact sorted).