universität innsbruck

WS 2018/2019

Functional Programming

Lecture 7

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Topics

abstract data types, algebraic data types, binary search trees, combinator parsing, efficiency, encoding data types as lambda-terms, evaluation strategies, formal verification, first steps, guarded recursion, Haskell introduction, higher-order functions, historical overview, induction, infinite data structures, input and output, lambda-calculus, lazy evaluation, list comprehensions, lists, modules, pattern matching, polymorphism, property-based testing, reasoning about functional programs, recursive functions, sets, strings, tail recursion, trees, tupling, type checking, type inference, types, types and type classes, unification, user-defined types

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Overview

- Mathematical Induction
- Induction over Lists
- Structural Induction
- Formal Verification of Functional Programs

Mathematical Induction

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show property for 0

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assume $P(k)$ (induction hypothesis), show $P(k+1)$

Why does this Work?

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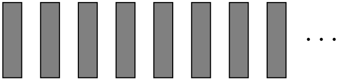

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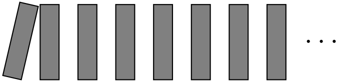

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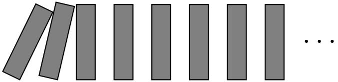

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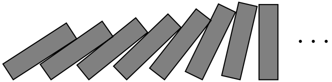

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Induction Principle

$$(P(m) \wedge \forall k \geq m.\, (P(k) \longrightarrow P(k+1))) \longrightarrow \forall n \geq m.\, P(n)$$

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&\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

Induction over Lists

Algebraic Data Type of Lists

```
data [a] = [] | (:) a [a]
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data [a] = [] | (:) a [a]
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Notes

- lists are recursive structures
- non-recursive constructor (base case): `[]`
- recursive constructor (step case): $x : xs$

Induction Principle for Lists – Informally

- to show $P(xs)$ for all lists xs
- show base case: $P(\texttt{[]})$
- show step case: $P(xs) \longrightarrow P(x \texttt{ : } xs)$ for arbitrary x and xs

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Remark

- for lists, P can be seen as function `p :: [a] -> Bool`

Exercise – Nil is right identity of append

- definition of append

```
   []    ++ ys  =  ys
(x:xs) ++ ys  =  x : (xs ++ ys)
```

- prove that [] is right identity of ++, that is,

$$xs \mathbin{+\!\!+} [] = xs$$

Exercise – Append is associative

- recall

```
  []   ++ ys  =  ys
(x:xs) ++ ys  =  x : (xs ++ ys)
```

- prove that ++ is associative, that is,

$$(xs +\!\!+ ys) +\!\!+ zs = xs +\!\!+ (ys +\!\!+ zs)$$

Exercise – Length and append

- definition

```
length []      =  0
length (_:xs)  =  1 + length xs
```

- prove that length of combined list is sum of lengths, that is,

$$\texttt{length}\ (xs \mathbin{+\!+} ys) = \texttt{length}\ xs + \texttt{length}\ ys$$

Structural Induction

Example – Terms

```
type Id = String
data Term = Var Id
          | App Term Term
          | Abs Id Term
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```haskell
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General Structures – Induction Principle

- for every non-recursive constructor, show base case
  - base case: $P(\texttt{Var x})$
- for every recursive constructor, show step case
  - step case 1: $(P(\texttt{s}) \wedge P(\texttt{t})) \longrightarrow P(\texttt{App s t})$
  - step case 2: $P(\texttt{t}) \longrightarrow P(\texttt{Abs x t})$

Example – Binary Trees

```
data BTree a = Empty
             | Node a (BTree a) (BTree a)
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Induction Principle for Binary Trees

$$(P(\texttt{Empty}) \wedge \forall x.\, \forall l.\, \forall r.\, ((P(l) \wedge P(r)) \longrightarrow P(\texttt{Node}\ x\ l\ r))) \longrightarrow \forall t.\, P(t)$$

Exercise – Perfect Binary Trees

- a binary tree is perfect if all leaf nodes have same depth

```
perfect Empty          = True
perfect (Node _ l r) =
  height l == height r && perfect l && perfect r

height Empty          = 0
height (Node _ l r) =
  max (height l) (height r) + 1

size Empty          = 0
size (Node _ l r) = size l + size r + 1
```

- lemma: a perfect binary tree t of height n has exactly $2^n - 1$ nodes, that is,

$$P(t) = \big(\texttt{perfect}\ t \longrightarrow \texttt{size}\ t = 2^{\texttt{height}\ t} - 1\big)$$

Formal Verification of Functional Programs

Isabelle/HOL in a Nutshell

Obvious question:

- What is Isabelle?

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Common answer:

- An LCF-style proof assistant.

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Typical follow-up questions:

- What is a proof assistant?
- What does LCF-style mean?
- ...

What is a Proof Assistant?

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Example

- automatic methods: logical reasoning (`blast`), equational reasoning (`simp`), combination of former (`auto`), ...
- manual steps: induction (`induct`), case analysis (`cases`), ...

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Example

- functions `assume : cterm -> thm` and `implies_elim : thm -> thm -> thm`
- implement inference rules

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \Longrightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

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Example

certified term

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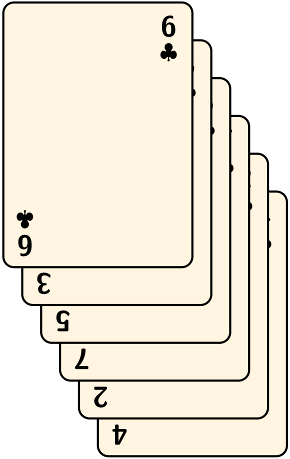
Higher-Order Logic

- HOL = Functional Programming + Logic
- data types (`datatype`)
- recursive functions (`fun`)
- logical operators ($\wedge$, $\vee$, $\longrightarrow$, $\forall$, $\exists$, ...)

Example – Insertion Sort



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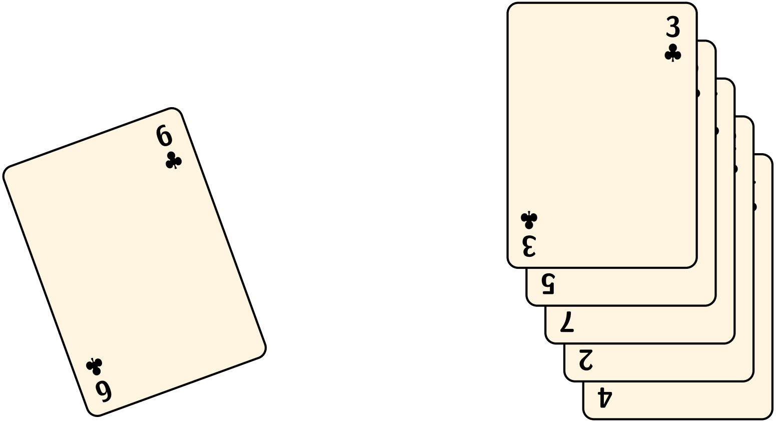

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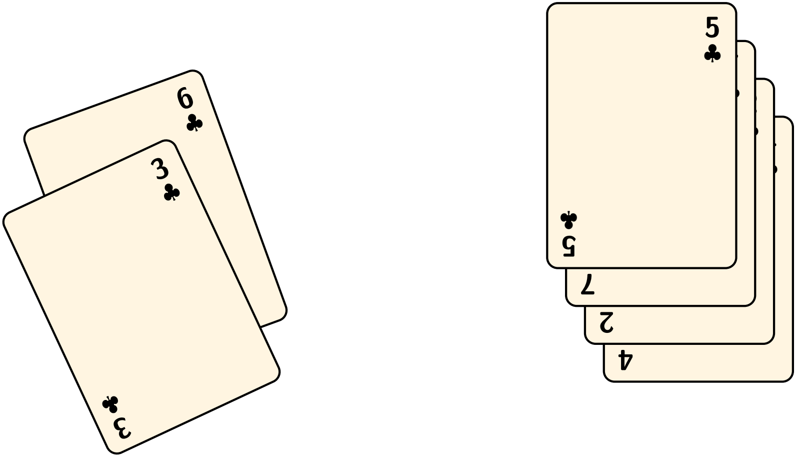

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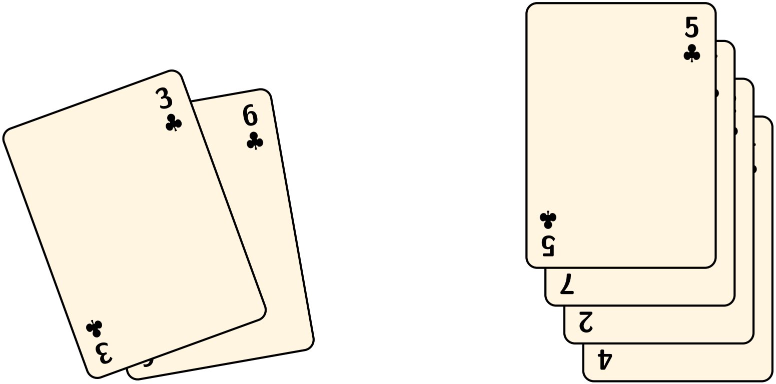

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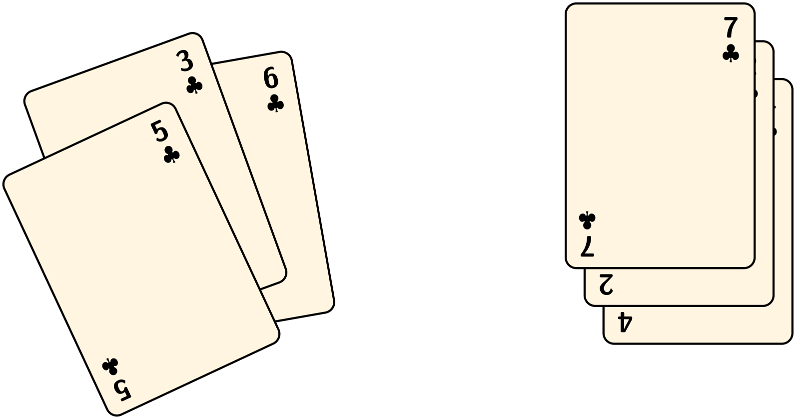

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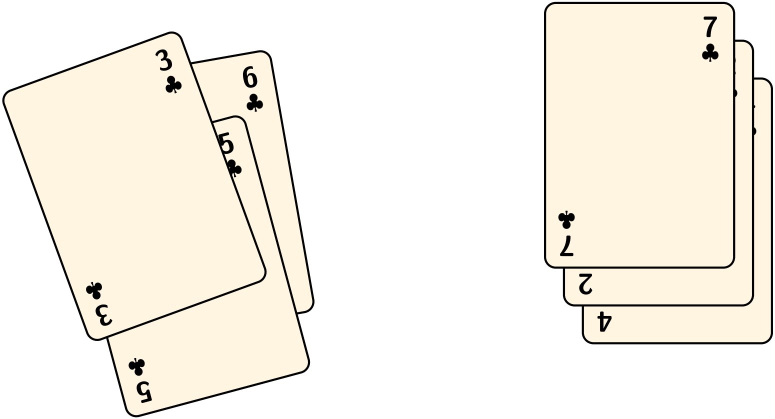

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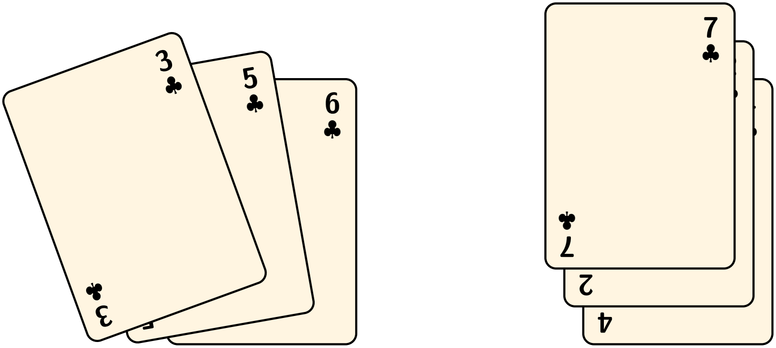

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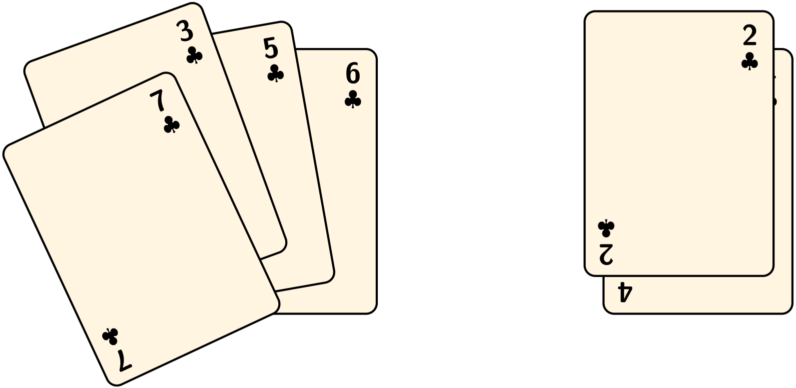

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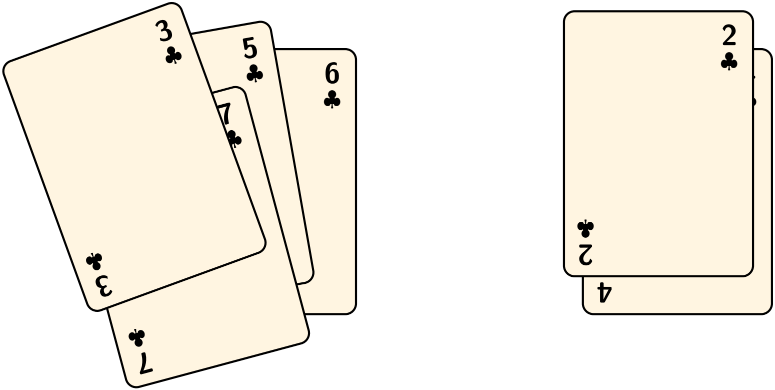

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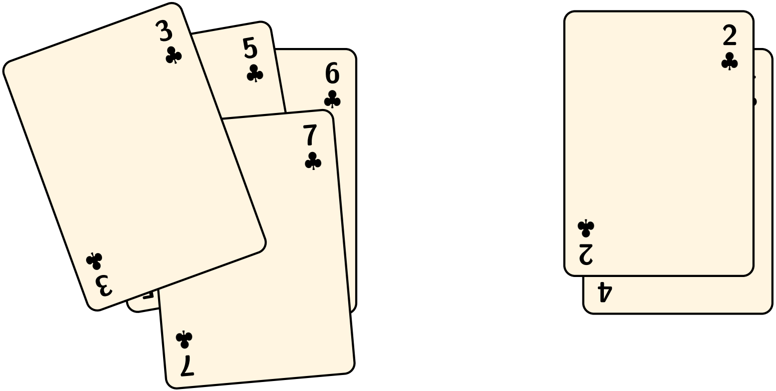

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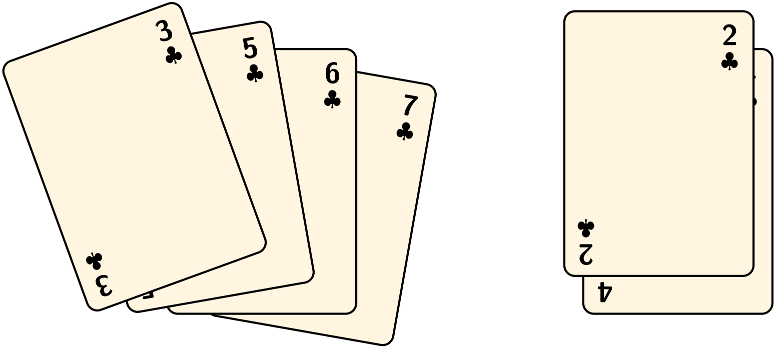

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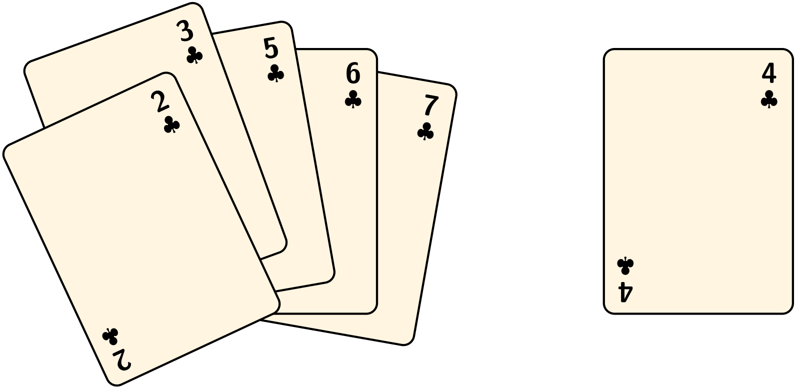

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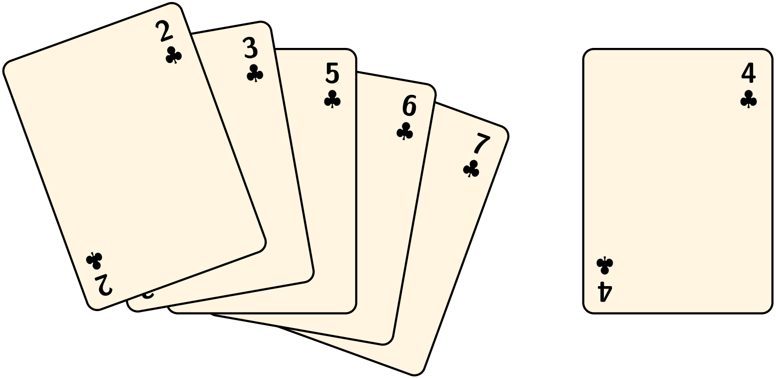

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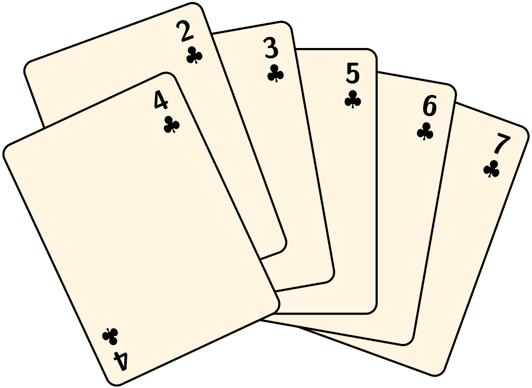

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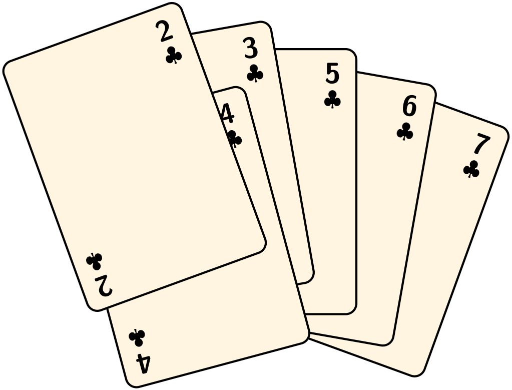

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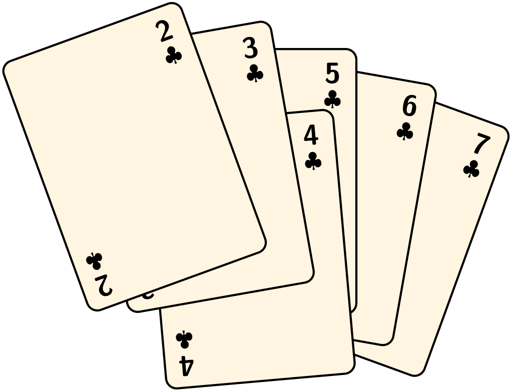

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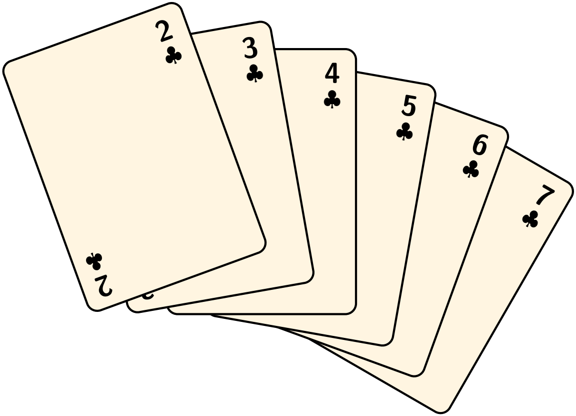

A Functional Implementation

- inserting an element into a sorted list

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insert x [] = [x]
insert x (y:ys) =
  if x <= y then x : y : ys
  else y : insert x ys
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- see `Insertion_Sort.thy`

Homework (for December 14th)

1. Read the lecture notes on reasoning about functional programs.
2. Prove $\texttt{map}\ f\ (\texttt{map}\ g\ xs) = \texttt{map}\ (f \circ g)\ xs$ for
```
map f [] = []
map f (x:xs) = f x : map f xs
```
3. Prove $\texttt{filter}\ p\ (\texttt{map}\ f\ xs) = \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ xs)$ for
```
filter p [] = []
filter p (x:xs) | p x = x : filter p xs
                | otherwise = filter p xs
```
4. Prove $\texttt{map}\ f\ (xs \mathbin{++} ys) = \texttt{map}\ f\ xs \mathbin{++} \texttt{map}\ f\ ys$ for
```
[] ++ ys = ys
(x:xs) ++ ys = x : (xs ++ ys)
```
5. Prove $\forall xs.\ \texttt{take}\ n\ (\texttt{map}\ f\ xs) = \texttt{map}\ f\ (\texttt{take}\ n\ xs)$ for
```
take n (x:xs) | n > 0 = x : take (n - 1) xs
take _ _ = []
```

Homework (for December 14th, continued)

6. Prove $\forall xs.\, \texttt{take}\ n\ xs \mathbin{++} \texttt{drop}\ n\ xs = xs$ for

```
drop n (_:xs) | n > 0 = drop (n - 1) xs
drop _ xs = xs
```

Alternatively

Choose two of the previous exercises and prove them with Isabelle/HOL using the custom type

```
datatype 'a lst = NIL | CONS 'a "'a lst"
```

and your own implementations of the relevant functions among

```
map :: "('a => 'b) => 'a lst => 'b lst"
filter :: "('a => bool) => 'a lst => 'a lst"
app :: "'a lst => 'a lst => 'a lst"
take :: "nat => 'a lst => 'a lst"
drop :: "nat => 'a lst => 'a lst"
```