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Functional Programming

Lecture 7

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Topics

abstract data types, algebraic data types, binary search trees, combinator parsing, efficiency, encoding data types as lambda-terms, evaluation strategies, formal verification, first steps, guarded recursion, Haskell introduction, higher-order functions, historical overview, induction, infinite data structures, input and output, lambda-calculus, lazy evaluation, list comprehensions, lists, modules, pattern matching, polymorphism, property-based testing, reasoning about functional programs, recursive functions, sets, strings, tail recursion, trees, tupling, type checking, type inference, types, types and type classes, unification, user-defined types

Overview

- Mathematical Induction
- Induction over Lists
- Structural Induction
- Formal Verification of Functional Programs

When to use Mathematical Induction?

- prove some property P for all natural numbers
- more formally, prove:

$$\forall n.\, P(n) \qquad \text{(where } n \in \mathbb{N}\text{)}$$

How is it Applied?

- mathematical induction consists of two steps:
  1. prove base case

     $$P(0)$$

     show property for 0

  2. prove step case

     $$\forall k.\, (P(k) \longrightarrow P(k+1))$$

     assume $P(k)$ (induction hypothesis), show $P(k+1)$

Why does this Work?

- have two facts:
  1. P true for 0
  2. for arbitrary k, if P true for k then P true for $k+1$
- want to show P for every natural number ($\forall n.\, P(n)$)

Example – $P(3)$

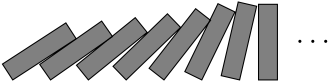
- have $P(0)$
- and $P(0) \longrightarrow P(1)$
- thus $P(1)$
- with $P(1) \longrightarrow P(2)$
- have $P(2)$
- with $P(2) \longrightarrow P(3)$
- have $P(3)$

Idea

- reach arbitrary n s.t. $P(n)$
- hence, $\forall n.\, P(n)$

Domino Effect

1. first domino falls
2. if domino falls, right neighbor falls



What is a “Property”?

- anything that depends on some input and is either true or false
- that is, some function `p :: a -> Bool`

Remark

- base case may be changed
- e.g., if base case $P(1)$, property holds for all $n \geq 1$

Induction Principle

$$(P(m) \wedge \forall k \geq m.\, (P(k) \longrightarrow P(k+1))) \longrightarrow \forall n \geq m.\, P(n)$$

Example – Gauß's Formula

- $P(x) = (1 + 2 + \cdots + x = \frac{x(x+1)}{2})$
- base case: $P(0) = (1 + 2 + \cdots + 0 = 0 = \frac{0(0+1)}{2})$
- step case: $P(k) \longrightarrow P(k+1)$
  IH: $P(k) = (1 + 2 + \cdots + k = \frac{k(k+1)}{2})$
  show: $P(k+1)$

$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Algebraic Data Type of Lists

```
data [a] = [] | (:) a [a]
```

Notes

- lists are recursive structures
- non-recursive constructor (base case): `[]`
- recursive constructor (step case): x : xs

Induction Principle for Lists – Informally

- to show $P(xs)$ for all lists xs
- show base case: $P(\texttt{[]})$
- show step case: $P(xs) \longrightarrow P(x : xs)$ for arbitrary x and xs

Induction Principle for Lists – Formally

$$(P(\texttt{[]}) \wedge \forall x. \forall xs. (P(xs) \longrightarrow P(x : xs))) \longrightarrow \forall xs. P(xs)$$

Remark

- for lists, P can be seen as function `p :: [a] -> Bool`

Exercise – Nil is right identity of append

- definition of append

```
   []    ++ ys  =  ys
(x:xs)  ++ ys  =  x : (xs ++ ys)
```

- prove that `[]` is right identity of ++, that is,

$$xs \mathbin{++} [] = xs$$

Exercise – Append is associative

- recall

```
  []    ++ ys  =  ys
(x:xs) ++ ys  =  x : (xs ++ ys)
```

- prove that ++ is associative, that is,

$$(xs \mathbin{+\!+} ys) \mathbin{+\!+} zs = xs \mathbin{+\!+} (ys \mathbin{+\!+} zs)$$

Exercise – Length and append

- definition

```
length []      =  0
length (_:xs)  =  1 + length xs
```

- prove that length of combined list is sum of lengths, that is,

$$\texttt{length}\ (xs \mathbin{+\!+} ys) = \texttt{length}\ xs + \texttt{length}\ ys$$

Example – Terms

```
type Id = String
data Term = Var Id
          | App Term Term
          | Abs Id Term
```

General Structures – Induction Principle

- for every non-recursive constructor, show base case
  - base case: $P(\texttt{Var x})$
- for every recursive constructor, show step case
  - step case 1: $(P(\texttt{s}) \wedge P(\texttt{t})) \longrightarrow P(\texttt{App s t})$
  - step case 2: $P(\texttt{t}) \longrightarrow P(\texttt{Abs x t})$

Example – Binary Trees

```
data BTree a = Empty
             | Node a (BTree a) (BTree a)
```

Induction Principle for Binary Trees

$$(P(\texttt{Empty}) \land \forall x. \forall l. \forall r. ((P(l) \land P(r)) \longrightarrow P(\texttt{Node}\ x\ l\ r))) \longrightarrow \forall t. P(t)$$

Exercise – Perfect Binary Trees

- a binary tree is perfect if all leaf nodes have same depth

```
perfect Empty        = True
perfect (Node _ l r) =
  height l == height r && perfect l && perfect r

height Empty        = 0
height (Node _ l r) =
  max (height l) (height r) + 1

size Empty        = 0
size (Node _ l r) = size l + size r + 1
```

- lemma: a perfect binary tree t of height n has exactly $2^n - 1$ nodes, that is,

$$P(t) = \left(\texttt{perfect}\ t \longrightarrow \texttt{size}\ t = 2^{\texttt{height}\ t} - 1\right)$$

Isabelle/HOL in a Nutshell

Obvious question:

- What is Isabelle?

Common answer:

- An LCF-style proof assistant.

Typical follow-up questions:

- What is a proof assistant?
- What does LCF-style mean?
- ...

What is a Proof Assistant?

- combination of automated theorem prover (ATP) and proof checker
- some subproofs are found automatically
- others are user-supplied but checked rigorously

Example

- automatic methods: logical reasoning (`blast`), equational reasoning (`simp`), combination of former (`auto`), ...
- manual steps: induction (`induct`), case analysis (`cases`), ...

What does LCF-style mean?

- theorems represented by abstract data type (`thm`)
- set of (basic) logical inferences provided as interface (trusted kernel)
- strong typing guarantees that there is no other way to create theorems (values of type `thm`)

Example

certified term

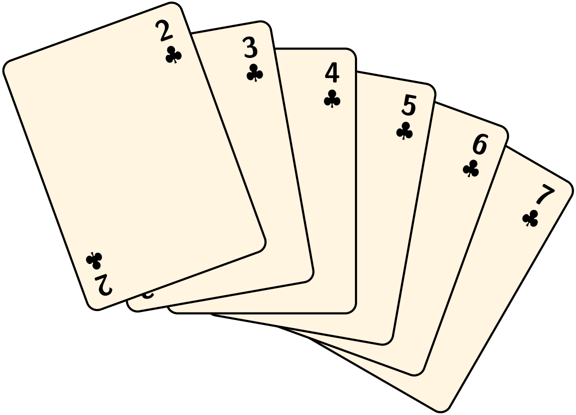
- functions `assume : cterm -> thm` and `implies_elim : thm -> thm -> thm`
- implement inference rules

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \Longrightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

Higher-Order Logic

- HOL = Functional Programming + Logic
- data types (`datatype`)
- recursive functions (`fun`)
- logical operators ($\wedge$, $\vee$, $\longrightarrow$, $\forall$, $\exists$, $\ldots$)

Example – Insertion Sort



A Functional Implementation

- inserting an element into a sorted list

```
insert x [] = [x]
insert x (y:ys) =
  if x <= y then x : y : ys
  else y : insert x ys
```

- sorting by repeatedly inserting elements into the empty list

```
insertionSort = foldr insert []
```

Exercise – Insertion sort is a valid sorting algorithm

- prove that result after applying insertion sort is sorted
- prove that all values occur exactly the same number of times in input and output
- see `Insertion_Sort.thy`

Homework (for December 14th)

1. Read the lecture notes on reasoning about functional programs.
2. Prove $\texttt{map}\ f\ (\texttt{map}\ g\ xs) = \texttt{map}\ (f \circ g)\ xs$ for

```
map f [] = []
map f (x:xs) = f x : map f xs
```

3. Prove $\texttt{filter}\ p\ (\texttt{map}\ f\ xs) = \texttt{map}\ f\ (\texttt{filter}\ (p \circ f)\ xs)$ for

```
filter p [] = []
filter p (x:xs) | p x = x : filter p xs
                | otherwise = filter p xs
```

4. Prove $\texttt{map}\ f\ (xs \mathbin{+\!\!+} ys) = \texttt{map}\ f\ xs \mathbin{+\!\!+} \texttt{map}\ f\ ys$ for

```
[] ++ ys = ys
(x:xs) ++ ys = x : (xs ++ ys)
```

5. Prove $\forall xs.\, \texttt{take}\ n\ (\texttt{map}\ f\ xs) = \texttt{map}\ f\ (\texttt{take}\ n\ xs)$ for

```
take n (x:xs) | n > 0 = x : take (n - 1) xs
take _ _ = []
```

Homework (for December 14th, continued)

6. Prove $\forall xs.\, \texttt{take}\ n\ xs +\!\!+ \texttt{drop}\ n\ xs = xs$ for

```
drop n (_:xs) | n > 0 = drop (n - 1) xs
drop _ xs = xs
```

Alternatively

Choose two of the previous exercises and prove them with Isabelle/HOL using the custom type

```
datatype 'a lst = NIL | CONS 'a "'a lst"
```

and your own implementations of the relevant functions among

```
map :: "('a => 'b) => 'a lst => 'b lst"
filter :: "('a => bool) => 'a lst => 'a lst"
app :: "'a lst => 'a lst => 'a lst"
take :: "nat => 'a lst => 'a lst"
drop :: "nat => 'a lst => 'a lst"
```