universität innsbruck

WS 2018/2019

Functional Programming

Lecture 8

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Topics

abstract data types, algebraic data types, binary search trees, combinator parsing, efficiency, encoding data types as lambda-terms, evaluation strategies, formal verification, first steps, guarded recursion, Haskell introduction, higher-order functions, historical overview, induction, infinite data structures, input and output, lambda-calculus, lazy evaluation, list comprehensions, lists, modules, pattern matching, polymorphism, property-based testing, reasoning about functional programs, recursive functions, sets, strings, tail recursion, trees, tupling, type checking, type inference, types, types and type classes, unification, user-defined types

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Overview

- Efficiency – Fibonacci Numbers
- Tupling
- Tail Recursion and Guarded Recursion
- Property-Based Testing with LeanCheck

Efficiency – Fibonacci Numbers

Definition – nth Fibonacci Number

$$\mathsf{fib}(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ \mathsf{fib}(n-1) + \mathsf{fib}(n-2) & \text{otherwise} \end{cases}$$

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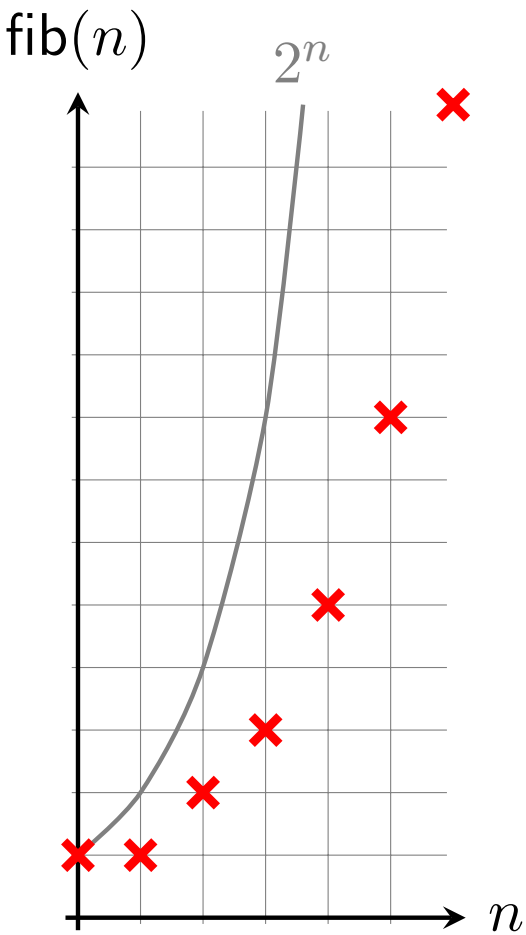
Graph



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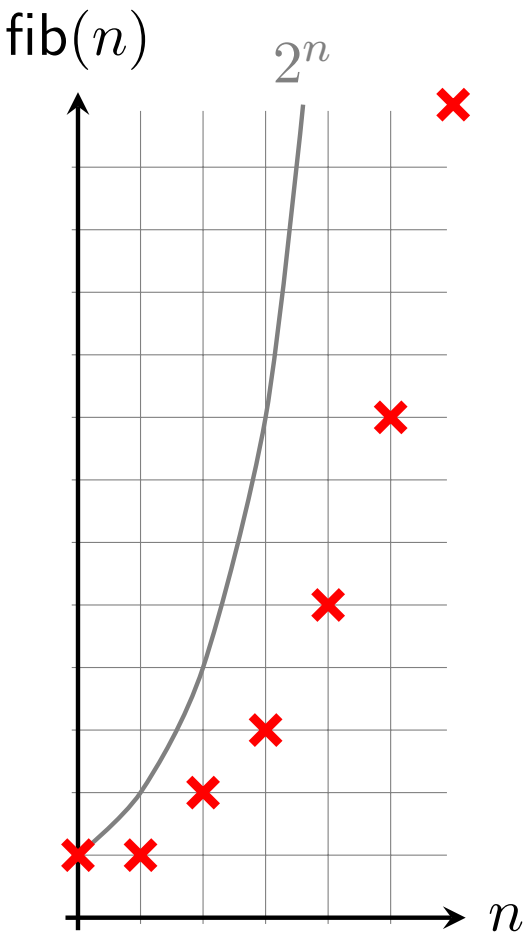

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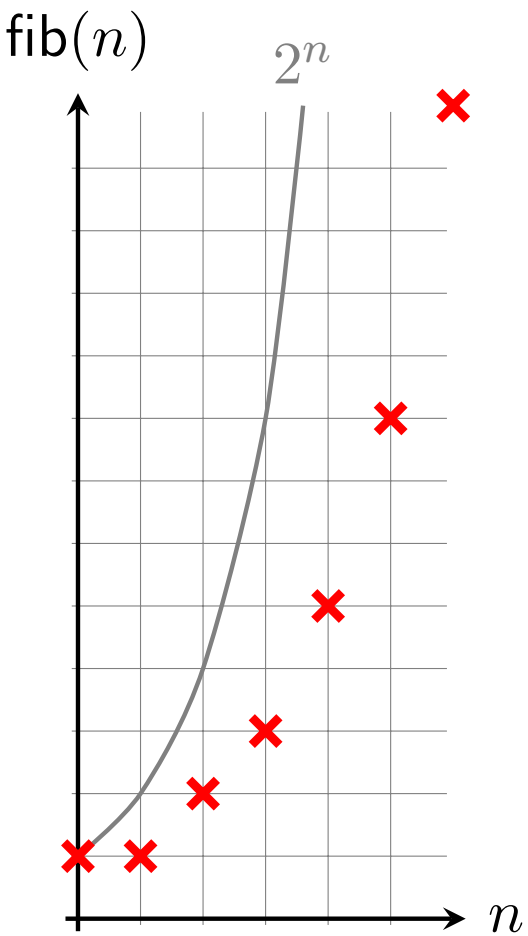

Example

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181 ,6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155, 165580141, 267914296, 433494437, 701408733, 1134903170, 1836311903, 2971215073, 4807526976, 7778742049, 12586269025, 20365011074, 32951280099, 53316291173, ...

Haskell Definition

```
fib n | n <= 1      =  1
      | otherwise   =  fib (n - 1) + fib (n - 2)
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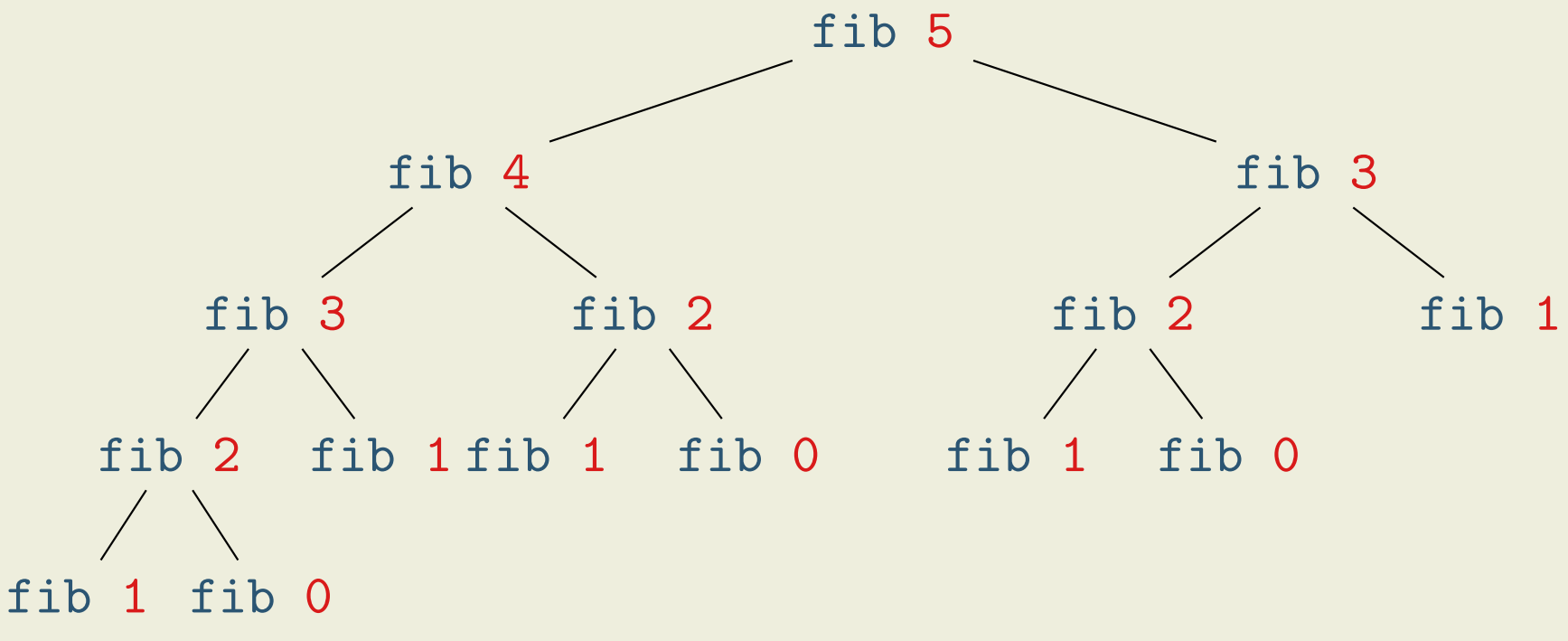

Tupling

Combining Results

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- make results available as return values instead of recomputing them

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Fibonacci Numbers – Alternative Definition

- definition

```
fib' = snd . fibpair
  where fibpair n | n <= 0     = (0, 1)
                  | otherwise = (f2, f1 + f2)
          where (f1, f2) = fibpair (n - 1)
```

- this function is linear in n
- since every recursive call reduces n by one

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```

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Exercise – fibpair computes fib

$$\texttt{fibpair}\ (n+1) = (\texttt{fib}\ n, \texttt{fib}\ (n+1))$$

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- combined function

```
average' xs = if l /= 0 then s / l else 0
  where
    (s, l)          = sumlen xs
    sumlen []       = (0, 0)
    sumlen (x:xs) = (s + x, l + 1)
      where
        (s, l) = sumlen xs
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- one traversal of `xs` suffices

Exercise

show `sumlen` $xs = (\texttt{sum}\ xs, \texttt{length}\ xs)$ by induction over xs

Tail Recursion and Guarded Recursion

Recursion vs. Tail Recursion

- a function calling itself is recursive

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```
length   []    = 0
length (x:xs) = 1 + length xs
```

Example – Mutually Recursive (and Tail Recursive)

```
even n | n <= 0     = True
       | otherwise = odd (n - 1)
odd n  | n <= 0     = False
       | otherwise = even (n - 1)
```

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Example – Guardedly Recursive

```
map f [] = []
map f (x:xs) = f x : map f xs
```

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- why? (tail recursive functions can be transformed into space-efficient loops automatically)

Example – Tail Recursive

```
reverse = rev []
  where
    rev acc []     = acc
    rev acc (x:xs) = rev (x:acc) xs
```

Example

```
sumlen' = sl 0 0
  where
    sl s l    []    = (s, l)
    sl s l (x:xs) = sl (s + x) (l + 1) xs
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Exercise

- prove that `sumlen'` $xs = (\texttt{sum}\ xs, \texttt{length}\ xs)$ for all xs

Problem with `sumlen'` – Memory Leak

- lazy evaluation
- hence `s+x` and `l+1` are only evaluated when result of `sumlen'` is used
- causes massive memory consumption
- e.g.,

  $$0 + 1 + 2 + \cdots + 100000000$$

  is stored for computing `sumlen' [1..100000000]`
- an unevaluated thunk of 100000001 integers (which requires more than 20GB of RAM)

Property-Based Testing with LeanCheck

Example – Grouping Consecutive Copies

- specification of `group`: turn given list into list of lists, such that order of elements is preserved and consecutive copies are packed together

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- implementation

```
group :: Eq a => [a] -> [[a]]
group [] = []
group [x] = [[x]]
group (x:y:ys) | x == y = [x,y] : group ys
               | otherwise = [x] : group (y:ys)
```

Unit Testing

- check program unit (for example, single function)
- check specific inputs (like `[]`, `[1,2,3]`, `[1,1,2,2]`)
  against expected results (like `[]`, `[[1],[2],[3]]`, `[[1,1],[2,2]]`

Unit Testing

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  against expected results (like `[]`, `[[1],[2],[3]]`, `[[1,1],[2,2]]`

Example – `group`

```
unitTests :: Bool
unitTests = and [
  group [] == ([] :: [[Int]]),
  group [1,2,3] == [[1],[2],[3]],
  group [1,1,2,2] == [[1,1],[2,2]]
  ]
```

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- concatenating groups results in original input

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prop_concat :: Eq a => [a] -> Bool
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Missing

- how to try inputs?
- possibilities: enumerate or random-generate

Installing LeanCheck with Cabal

- Cabal is system for building, packaging, and installing Haskell libraries and programs
- update package list

```
$ cabal update
```

- install latest Cabal libraries

```
$ cabal install Cabal
```

- install LeanCheck

```
$ cabal install leancheck
```

- now we can use LeanCheck

```
import Test.LeanCheck
```

- (tested on `zid-gpl.uibk.ac.at`)

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- `c ==> p` – implication, test `p` only if condition `c` holds

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- `check p` – check property `p` and generate report as IO action
- `c ==> p` – implication, test `p` only if condition `c` holds
- `list :: [a]` – (potentially infinite) enumeration of values of type `a` (these are the values used for testing)

Exercise

test `prop_concat` and `prop_nonempty`

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Example – More Properties of `group`

- all elements of a group are equal

```
prop_all_equal :: Eq a => [a] -> Bool
prop_all_equal xs = all all_equal (group xs)
  where
    all_equal (x:y:ys) = x == y && all_equal (y:ys)
    all_equal _ = True
```

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```

- check that consecutive groups do not contain same elements

```
prop_nonequal :: Eq a => [a] -> Bool
prop_nonequal xs = nonequal (group xs)
  where
    nonequal (x:y:ys) = head x /= head y &&
      nonequal (y:ys)
    nonequal _ = True
```

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Example – Output of Exercise

```
ghci> check prop_all_equal
+++ OK, passed 200 tests.
ghci> check prop_nonequal
*** Failed! Falsifiable (after 4 tests):
[(),(),()]
```

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Correct Implementation of `group`

```
group :: Eq a => [a] -> [[a]]
group [] = []
group [x] = [[x]]
group (x:y:ys) | x == y = (x:zs) : zss
               | otherwise = [x] : zs : zss
  where zs:zss = group (y:ys)
```

Examples

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- which values did we actually test?

```
ghci> witnesses 200 prop_rev_app
[["[]","[]"],["[]","[()]"],["[()]","[]"],
 ["[]","[(),()]"],["[()]","[()]"], ...]
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- without type annotations, type variables default to unit type `()`
- better

```
ghci> check (prop_rev_app :: [Int] -> [Int] -> Bool)
```

Examples (cont'd)

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- a conditional property

```
prop_take_neq i xs =
  i < length xs ==> take i xs /= xs
```

- does it hold?

Homework (for January 11th)

1. Read the lecture notes on efficiency, https://wiki.haskell.org/Tail_recursion, and http://en.wikipedia.org/wiki/Tail_recursion#Tail_recursion_modulo_cons
2. Find one function that is tail recursive, one that is guardedly recursive, and one that is neither, in the lecture slides of the earlier weeks. In each case, justify your answer.
3. Use tupling to implement a more efficient version of `initLast xs = (init xs, last xs)` and prove by induction that it coincides with `initLast` on non-empty lists.
4. Give a tail recursive implementation of the factorial function

   ```
   fac n | n <= 1    = 1
         | otherwise = n * fac (n - 1)
   ```

   and prove by induction that it coincides with `fac`.

Homework (for January 11th, continued)

5. Use LeanCheck to test whether

```
sort' xs = foldr insort [] xs
  where insort x [] = [x]
        insort x (y:ys)
          | x <  y    = x:y:ys
          | x == y    = x:ys
          | otherwise = y : insort x ys
```

   is a correct sorting function.
   Hint: Implement two functions `sorted :: Ord a => [a] -> Bool` and `count :: Eq a => [a] -> a -> Int` and use them to express the desired property.

6. Give a tail recursive variant of

```
nfold n f x = head $ drop n $ iterate f x
```

   and use LeanCheck to test whether it coincides with `nfold`.
   Hint: Import `Test.LeanCheck.Function` to test properties of higher-order functions.

Optional Programming Exercise (for January 11th)

Setup:

- (this exercise is worth up to 6 crosses)
- consider propositional formulas $\phi ::= \bot \mid p \mid \phi \to \phi$
- additional logical connectives given by $\neg\phi \stackrel{\text{def}}{=} \phi \to \bot$, $\phi \vee \psi \stackrel{\text{def}}{=} \neg\phi \to \psi$, and $\phi \wedge \psi \stackrel{\text{def}}{=} \neg(\phi \to \neg\psi)$
- six basic inference rules

$$\frac{}{\phi \vdash \phi}\ (\text{trivial}) \qquad \frac{\Gamma \vdash \phi}{\psi, \Gamma \vdash \phi}\ (\text{assume}) \qquad \frac{\Gamma, \phi, \Delta \vdash \psi \quad \phi \in \Gamma \cup \Delta}{\Gamma, \Delta \vdash \psi}\ (\text{discard})$$

$$\frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}\ (\neg\neg\text{e}) \qquad \frac{\Gamma, \phi, \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \to \psi}\ (\to\text{i}) \qquad \frac{\Gamma \vdash \phi \to \psi \quad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}\ (\to\text{e})$$

Optional Programming Exercise (for January 11th, continued)

- Based on the module `Formula`, implement an ADT `Proof` together with the six basic inference rules represented by the Haskell functions:

```
trivial :: Formula -> Proof
assume  :: Formula -> Proof -> Proof
discard :: Int -> Proof -> Maybe Proof
impI    :: Int -> Proof -> Maybe Proof
impE    :: Proof -> Proof -> Maybe Proof
nnegE   :: Proof -> Maybe Proof
```

- Furthermore, implement as many of the following derived rules as possible without breaking the abstraction:

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \neg\neg\phi}\ (\neg\neg\mathsf{i}) \qquad \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \wedge \psi}\ (\wedge\mathsf{i}) \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2 \quad i \in \{1,2\}}{\Gamma \vdash \phi_i}\ (\wedge\mathsf{e})$$

$$\frac{\Gamma \vdash \phi_i \quad i \in \{1,2\}}{\Gamma \vdash \phi_1 \vee \phi_2}\ (\vee\mathsf{i}) \qquad \frac{\Gamma \vdash \phi \vee \psi \quad \phi, \Gamma \vdash \chi \quad \psi, \Gamma \vdash \chi}{\Gamma \vdash \chi}\ (\vee\mathsf{e})$$

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
```

```
> peirce (read "A") (read "B")
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
```

```
> peirce (read "A") (read "B")
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
```

```
> peirce (read "A") (read "B")
A → ⊥, A ⊢ ⊥
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
```

```
> peirce (read "A") (read "B")
A → ⊥, A ⊢ (B → ⊥) → ⊥
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
```

```
> peirce (read "A") (read "B")
A → ⊥, A ⊢ B
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
  let Just p4 = trivial ((f `Imp` g) `Imp` f) `impE` p3 in
```

```
> peirce (read "A") (read "B")
A → ⊥ ⊢ A → B
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
  let Just p4 = trivial ((f `Imp` g) `Imp` f) `impE` p3 in
  let Just p5 = trivial (f `Imp` Bot) `impE` p4 in
```

```
> peirce (read "A") (read "B")
(A → B) → A, A → ⊥ ⊢ A
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
  let Just p4 = trivial ((f `Imp` g) `Imp` f) `impE` p3 in
  let Just p5 = trivial (f `Imp` Bot) `impE` p4 in
  let Just p6 = discard 0 p5 in
```

```
> peirce (read "A") (read "B")
A → ⊥, (A → B) → A, A → ⊥ ⊢ ⊥
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
  let Just p4 = trivial ((f `Imp` g) `Imp` f) `impE` p3 in
  let Just p5 = trivial (f `Imp` Bot) `impE` p4 in
  let Just p6 = discard 0 p5 in
  let Just p7 = impI 1 p6 in
```

```
> peirce (read "A") (read "B")
(A → B) → A, A → ⊥ ⊢ ⊥
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
  let Just p4 = trivial ((f `Imp` g) `Imp` f) `impE` p3 in
  let Just p5 = trivial (f `Imp` Bot) `impE` p4 in
  let Just p6 = discard 0 p5 in
  let Just p7 = impI 1 p6 in
  let Just p8 = nnegE p7 in
```

```
> peirce (read "A") (read "B")
(A → B) → A ⊢ (A → ⊥) → ⊥
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
  let Just p4 = trivial ((f `Imp` g) `Imp` f) `impE` p3 in
  let Just p5 = trivial (f `Imp` Bot) `impE` p4 in
  let Just p6 = discard 0 p5 in
  let Just p7 = impI 1 p6 in
  let Just p8 = nnegE p7 in
  let Just p9 = impI 0 p8 in


> peirce (read "A") (read "B")
(A → B) → A ⊢ A
```

Example – Peirce's Law

```
peirce :: Formula -> Formula -> Proof
peirce f g =
  let Just p0 = trivial (f `Imp` Bot) `impE` trivial f in
  let Just p1 = impI 0 $ assume (g `Imp` Bot) p0 in
  let Just p2 = nnegE p1 in
  let Just p3 = impI 1 p2 in
  let Just p4 = trivial ((f `Imp` g) `Imp` f) `impE` p3 in
  let Just p5 = trivial (f `Imp` Bot) `impE` p4 in
  let Just p6 = discard 0 p5 in
  let Just p7 = impI 1 p6 in
  let Just p8 = nnegE p7 in
  let Just p9 = impI 0 p8 in
  p9
```

```
> peirce (read "A") (read "B")
 ⊢ ((A → B) → A) → A
```